

HIGHER COMPARISON MAPS FOR THE SPECTRUM OF A TENSOR TRIANGULATED CATEGORY

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ABSTRACT. For each object in a tensor triangulated category, we construct a natural continuous map from the object’s support—a closed subset of the category’s triangular spectrum—to the Zariski spectrum of a certain commutative ring of endomorphisms. When applied to the unit object this recovers a construction of P. Balmer. These maps provide an iterative approach for understanding the spectrum of a tensor triangulated category by starting with the comparison map for the unit object and iteratively analyzing the fibers of this map via “higher” comparison maps. We illustrate this approach for the stable homotopy category of finite spectra. In fact, the same underlying construction produces a whole collection of new comparison maps, including maps associated to (and defined on) each closed subset of the triangular spectrum. These latter maps provide an alternative strategy for analyzing the spectrum by iteratively building a filtration of closed subsets by pulling back filtrations of affine schemes.

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INTRODUCTION

Many triangulated categories arising in nature come equipped with natural \otimes -product structures—that is, they are *tensor* triangulated categories—and in recent years there has been a growing appreciation for the significance of these \otimes -structures. For example, a (nice) scheme can be recovered from the tensor triangulated structure of its derived category of perfect complexes, but not from the triangulated structure alone (see [Bal10a, Remark 64], for example). Using the \otimes -structure, Paul Balmer [Bal05] has introduced the *spectrum* of a tensor triangulated category. Just as the spectrum of a commutative ring provides a geometric

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approach to commutative algebra, the spectrum of a tensor triangulated category provides a geometric approach to the study of tensor triangulated categories—an approach referred to as *tensor triangular geometry* by its originators. The present paper makes a contribution to tensor triangular geometry and the attentive reader is referred to [Bal10b] for an introduction to this relatively new field and for additional background that leads to the present work.

Determining the spectrum of a given tensor triangulated category is a highly non-trivial problem, which is essentially equivalent to classifying the thick triangulated \otimes -ideals in the category—in other words, classifying the objects of the category up to the naturally available structure: \otimes -products, \oplus -sums, \oplus -summands, suspensions, and cofibers. Major classification theorems in algebraic geometry, modular representation theory and stable homotopy theory give complete descriptions of the spectrum in several important examples, but one of the goals of tensor triangular geometry is to go the other way—to develop techniques for determining the spectrum (and thereby solve the classification problem), or to at least say something interesting about the spectrum when a full determination proves to be too ambitious.

In any tensor triangulated category, the endomorphism ring of the unit is commutative, and the first step towards saying something about the spectrum of a general tensor triangulated category was taken in [Bal10a] where continuous maps

$$\rho : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}(\mathrm{End}_{\mathcal{K}}(\mathbb{1})) \quad \text{and} \quad \rho^\bullet : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}^h(\mathrm{End}_{\mathcal{K}}^\bullet(\mathbb{1}))$$

were defined going from the triangular spectrum to the (homogeneous) spectrum of the (graded) endomorphism ring of the unit. These “comparison maps” are often surjective and so attention focusses on understanding conditions under which they are injective and more generally on understanding their fibers. If $\mathcal{K} = D^{\mathrm{perf}}(A)$ is the derived category of perfect complexes of a commutative ring A , then $\mathrm{End}_{\mathcal{K}}(\mathbb{1})$ is isomorphic to A and ρ turns out to be a homeomorphism. This can be proved directly and provides an alternative proof of the affine case of the Hopkins-Neeman-Thomason theorem. On the other hand, if G is a finite group, k is a field, and $\mathcal{K} = D^b(kG\text{-mod})$ with $\otimes = \otimes_k$, then $\mathrm{End}_{\mathcal{K}}^\bullet(\mathbb{1})$ is group cohomology $H^\bullet(G, k)$ and it is known using the classification theorem of Benson-Carlson-Rickard that the map ρ^\bullet is a homeomorphism. A more direct proof of the injectivity of ρ^\bullet in this example would provide a new proof of the Benson-Carlson-Rickard theorem. In general, however, one cannot expect the (graded) endomorphisms of the unit to determine the global structure of the whole category and we are left with the important general problem of understanding the fibers of these comparison maps.

In this paper, we will construct new comparison maps which generalize those mentioned above. More specifically, for each object X in a tensor triangulated category \mathcal{K} we will define maps

$$\rho_X : \mathrm{supp}(X) \rightarrow \mathrm{Spec}(R_X) \quad \text{and} \quad \rho_X^\bullet : \mathrm{supp}(X) \rightarrow \mathrm{Spec}^h(R_X^\bullet)$$

from the support of X (a closed subset of the triangular spectrum) to the (homogeneous) spectrum of a certain (graded-)commutative ring of (graded) endomorphisms of X , which recover the original comparison maps when $X = \mathbb{1}$. The author’s initial interest in these new comparison maps stems from the fact that they provide a method for studying the fibers of the original maps. This in turn leads to an iterative strategy for studying the spectrum based on a repeated analysis of the fibers of a sequence of generalized comparison maps. The idea runs as follows.

Given an arbitrary tensor triangulated category \mathcal{K} , we can take the unit object and consider the comparison map $\rho_{\mathbb{1}} : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}(R_{\mathbb{1}})$. Understanding the fibers of this map reduces by a localization technique to the case when $R_{\mathbb{1}}$ is a local ring. If the unique closed point $\mathfrak{m} = \langle f_1, \dots, f_n \rangle$ is finitely generated then it is straightforward to show that the fiber $\rho_{\mathbb{1}}^{-1}(\{\mathfrak{m}\})$ is equal to the support of the object $X_{\mathbb{1}} := \mathrm{cone}(f_1) \otimes \cdots \otimes \mathrm{cone}(f_n)$. This fiber can then be examined more closely by considering the “higher” comparison map $\rho_{X_1} : \mathrm{supp}(X_1) \rightarrow \mathrm{Spec}(R_{X_1})$ associated with the object X_1 . The same procedure can then be used to study the fibers of ρ_{X_1} and the process repeats itself. Following any particular thread in this process produces a linear filtration

$$\mathrm{Spc}(\mathcal{K}) \supset \mathrm{supp}(X_1) \supset \mathrm{supp}(X_2) \supset \cdots \supset \mathrm{supp}(X_n)$$

which can be extended for however long the rings involved possess finitely generated primes.

One of the difficulties with this method is that to understand the fiber over a non-closed point we must first apply a localization procedure. The reason is that for a finitely generated prime $\mathfrak{p} = \langle f_1, \dots, f_n \rangle$, the support of $\mathrm{cone}(f_1) \otimes \cdots \otimes \mathrm{cone}(f_n)$ is actually the preimage of the closure $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ rather than the fiber over \mathfrak{p} . More generally, $\rho_X^{-1}(V(I)) = \mathrm{supp}(\mathrm{cone}(f_1) \otimes \cdots \otimes \mathrm{cone}(f_n))$ for any finitely generated ideal $I = \langle f_1, \dots, f_n \rangle$. Thus, rather than examining the fibers of a comparison map ρ_X , an alternative strategy is to take a look at the preimages of all of the Thomason closed subsets $V(I) \subset \mathrm{Spec}(R_X)$.¹ Choosing generators of the ideal I provides us with an object of \mathcal{K} whose support is the closed subset $\rho_X^{-1}(V(I))$ and we can examine this subset further via the comparison map associated with this “generator” object.²

Both of these strategies suffer from the fact that (a) they only deal with finitely generated primes and Thomason closed subsets (which may be an undesirable limitation in non-noetherian situations) and (b) they involve non-canonical choices of generators. The fundamental idea in both approaches is to examine a Thomason closed subset $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$ by the comparison map associated with an object which generates \mathcal{Z} , but this comparison map depends on the choice of generator. Such considerations lead to the desire for a “generator-independent” comparison map which only depends on the Thomason closed subset on which it is defined, and more generally for a comparison map associated to *every* closed subset of the spectrum.

Indeed, the map ρ_X is just one of a host of new comparison maps introduced in this paper. The most general construction associates a natural, continuous map

$$\rho_{\Phi} : \bigcap_{X \in \Phi} \mathrm{supp}(X) \rightarrow \mathrm{Spec}(R_{\Phi})$$

to each set of objects $\Phi \subset \mathcal{K}$ that is closed under the \otimes -product. Taking $\Phi = \{X^{\otimes n} \mid n \geq 1\}$ gives the map ρ_X above, while taking $\Phi = \{a \in \mathcal{K} \mid \mathrm{supp}(a) \supset \mathcal{Z}\}$ gives a map $\rho_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathrm{Spec}(R_{\mathcal{Z}})$ associated to (and defined on) an arbitrary closed subset of the spectrum. Following on from the previous discussion, the latter “closed

¹Recall that a Thomason closed subset is the same thing as a closed subset whose complement is quasi-compact. In the case of an affine scheme $\mathrm{Spec}(A)$ this is a closed set of the form $V(I)$ for a finitely generated ideal $I \subset A$, while in the case of $\mathrm{Spc}(\mathcal{K})$ this is a closed subset of the form $\mathrm{supp}(a)$ for an object $a \in \mathcal{K}$. These notions will be reviewed in Section 2.

²A “generator” of a closed subset $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$ is an object $a \in \mathcal{K}$ with $\mathrm{supp}(a) = \mathcal{Z}$.

set” comparison maps ρ_Z afford perhaps the most robust strategy for studying the spectrum. The idea is to iteratively build a filtration of closed subsets by pulling back filtrations of the affine schemes $\mathrm{Spec}(R_Z)$. This idea has the advantage that it utilizes all closed subsets (not just Thomason ones) and is purely deterministic: no choices are involved. The hope is that ultimately the filtration will become fine enough to completely determine the spectrum. Although certain difficulties prevent these strategies from working out in the full generality that one might hope, the author nevertheless considers them to be the primary justification for the theory developed in this paper.

In any case, one example where none of the difficulties arise is the stable homotopy category of finite spectra $\mathrm{SH}^{\mathrm{fin}}$. This is an elusive example for tensor triangular geometry. Although the structure of the space $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$ is known via the work of Devinatz, Hopkins and Smith (see [HS98] and [Bal10a, Section 9]), the unit comparison map

$$\begin{array}{ccccccc}
 & & \mathcal{C}_{2,\infty} & & \mathcal{C}_{3,\infty} & & \cdots & & \mathcal{C}_{p,\infty} & & \cdots \\
 & & | & & | & & & & | & & \\
 \mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}) = & & \vdots & & \vdots & & & & \vdots & & \\
 & & | & & | & & & & | & & \\
 & & \mathcal{C}_{2,n+1} & & \mathcal{C}_{3,n+1} & & \cdots & & \mathcal{C}_{p,n+1} & & \cdots \\
 & & | & & | & & & & | & & \\
 & & \mathcal{C}_{2,n} & & \mathcal{C}_{3,n} & & \cdots & & \mathcal{C}_{p,n} & & \cdots \\
 & & | & & | & & & & | & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & | & & | & & & & | & & \\
 & & \mathcal{C}_{2,2} & & \mathcal{C}_{3,2} & & \cdots & & \mathcal{C}_{p,2} & & \cdots \\
 & & \searrow & & \searrow & & & & \searrow & & \\
 & & & & & & & & \mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}} & & \\
 & & \searrow & & \searrow & & & & \searrow & & \\
 \mathrm{Spec}(\mathbb{Z}) = & & 2\mathbb{Z} & & 3\mathbb{Z} & & \cdots & & p\mathbb{Z} & & \cdots \\
 & & \searrow & & \searrow & & & & \searrow & & \\
 & & & & & & & & (0) & &
 \end{array}$$

is far from injective and understanding the fibers (which are given by the Morava K -theories) is related to the important problem of understanding residue fields in tensor triangular geometry. In any case, the iterative procedure we have mentioned above works out very nicely in this example, and it provides one illustration of how the higher comparison maps can work out in practice. However, determining the comparison maps in this example—in particular, determining the structure of the rings R_X^\bullet —requires the full strength of the results in [HS98] on nilpotence and periodicity in stable homotopy theory. In particular, it presupposes knowledge of the classification of thick subcategories in $\mathrm{SH}^{\mathrm{fin}}$. Nevertheless, these results allow us to show that the new comparison maps refine the view of $\mathrm{SH}^{\mathrm{fin}}$ provided by Balmer’s original comparison map $\rho_{\mathbb{1}}$. This is an important test for our theory as other generalizations of the original maps have failed to provide additional insight into this example.

The primary portion of this paper is devoted to the construction of the new comparison maps and laying the foundations of their basic theory. For example, we establish their naturality, show that passing to the idempotent completion does not change anything, and develop a technique for localizing with respect to primes in R_Φ (which has been alluded to in the discussion above and generalizes the “central localization” of [Bal10a]). Other results include establishing that the object comparison maps ρ_X are invariant under a number of natural operations that can be performed on the object X such as taking suspensions, or duals, or \otimes -powers, etc. In addition, we establish some connections of a topological nature between the target affine scheme $\text{Spec}(R_\Phi)$ and the domain of ρ_Φ ; for example, we show that the domain is connected if and only if $\text{Spec}(R_\Phi)$ is connected. Other results of that nature include establishing that the image of ρ_Φ is always dense in $\text{Spec}(R_\Phi)$.

There remain a number of unresolved questions and speculations related to the comparison maps defined in this paper and it will take more examples to determine the value of these constructions. Finally, it is worth mentioning that [DS] has also defined generalizations of the original comparison maps from [Bal10a]. However, that work focuses on invertible objects in the category and goes in quite a different direction than the present paper.

1. TENSOR TRIANGULATED CATEGORIES

A tensor triangulated category is a triangulated category (\mathcal{K}, Σ) together with a “compatible” symmetric monoidal structure $(\mathcal{K}, \otimes, \mathbb{1})$. More precisely, it is required that $- \otimes a : \mathcal{K} \rightarrow \mathcal{K}$ and $a \otimes - : \mathcal{K} \rightarrow \mathcal{K}$ be exact functors for each $a \in \mathcal{K}$. Implicit in this statement is the data of two natural isomorphisms

$$\Sigma a \otimes b \simeq \Sigma(a \otimes b) \quad \text{and} \quad a \otimes \Sigma b \simeq \Sigma(a \otimes b) \quad (1.1)$$

which relate the suspension and the tensor. In addition to requiring $a \otimes -$ and $- \otimes a$ be exact functors, these suspension isomorphisms are required to relate suitably with the symmetry, associator, and unitor isomorphisms of the symmetric monoidal structure (see [HPS97, Appendix A.2] for details). Finally, it is assumed that the diagram

$$\begin{array}{ccc} \Sigma a \otimes \Sigma b & \simeq & \Sigma(a \otimes \Sigma b) \\ \wr & & \wr \\ \Sigma(\Sigma a \otimes b) & \simeq & \Sigma^2(a \otimes b) \end{array} \quad (1.2)$$

anti-commutes.

By a morphism of tensor triangulated categories we mean a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ that is both an exact functor of triangulated categories as well as a strong \otimes -functor. Moreover, some compatibility is required between the various isomorphisms that are attached to the functor ($F\Sigma a \simeq \Sigma Fa$, $Fa \otimes Fb \simeq F(a \otimes b)$, $F\mathbb{1}_{\mathcal{C}} \simeq \mathbb{1}_{\mathcal{D}}$), as well as between those isomorphisms and the suspension isomorphisms of the categories \mathcal{C} and \mathcal{D} . The compatibility axioms are mostly obvious and are not usually spelled out. One point to be made is that the naturality of our graded comparison maps depends on the compatibility axiom which asserts that

$$\begin{array}{ccc} \Sigma(Fa \otimes Fb) & \simeq & \Sigma Fa \otimes Fb \simeq F\Sigma a \otimes Fb \\ \wr & & \wr \\ \Sigma F(a \otimes b) & \simeq & F\Sigma(a \otimes b) \simeq F(\Sigma a \otimes b) \end{array} \quad (1.3)$$

commutes.

Many authors include stronger assumptions about the monoidal structure in their definition of a tensor triangulated category. For example, it is common to assume that the monoidal structure is a *closed* symmetric monoidal structure (i.e., that internal homs exist). For the general constructions of this paper, we don't need anything more than a symmetric monoidal structure. However, for some results we will need to assume that our tensor triangulated category is *rigid*; in other words, that every object is dualizable and that the “taking duals” functor $D : \mathcal{K}^{\text{op}} \rightarrow \mathcal{K}$ preserves exact triangles. See [Bal10a, Definition 1.5] for a precise definition. Many tensor triangulated categories of interest are rigid, but we will be explicit about when this assumption is required.

Notation 1.4. For a collection of objects \mathcal{E} in a tensor triangulated category \mathcal{K} , $\langle \mathcal{E} \rangle$ will denote the thick \otimes -ideal generated by \mathcal{E} .

The reader is assumed to be familiar with the basic definitions and theory of the spectrum $\text{Spc}(\mathcal{K})$ of an essentially small tensor triangulated category \mathcal{K} (introduced in [Bal05]), as well as with the notion of a *local* tensor triangulated category (introduced in [Bal10a, Section 4]). In this paper, we will not explicitly state the assumption that \mathcal{K} is essentially small, but it will be tacitly assumed any time we speak of $\text{Spc}(\mathcal{K})$. Although the spectrum has the structure of a locally ringed space, for our purposes only its topological structure is relevant. Bear in mind that the Balmer topology on $\text{Spc}(\mathcal{K})$ is *not* the Zariski topology one would obtain by mimicking the definition of the Zariski topology on the prime spectrum of a commutative ring; from the point of view of spectral spaces (see below) it is the Hochster-dual of the Zariski topology. The result is that some things in tensor triangular geometry behave a bit differently than one might expect. For example, closure in the Balmer topology goes *down* rather than *up*: $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \text{Spc}(\mathcal{K}) \mid \mathcal{Q} \subset \mathcal{P}\}$. In particular, the closed points in $\text{Spc}(\mathcal{K})$ are the *minimal* primes. Another consequence of the differences between the Balmer and Zariski topologies is that our comparison maps will be inclusion-*reversing*.

2. SPECTRAL SPACES AND THOMASON SUBSETS

Definition 2.1. A topological space is *spectral* if it is T_0 , quasi-compact, the quasi-compact open subsets are closed under finite intersection and form an open basis, and every non-empty irreducible closed subset has a generic point.

Hochster [Hoc69] showed that a topological space is spectral if and only if it is homeomorphic to the Zariski spectrum of a commutative ring. On the other hand, the spectrum of a tensor triangulated category is spectral and it follows from the results of Hochster, Thomason, and Balmer that every spectral space arises in this way.

Definition 2.2. A subset $\mathcal{Y} \subset \mathcal{X}$ of a spectral space is *Thomason* if it is a union of closed subsets each of which has quasi-compact complement.

Hochster showed that every spectral space admits a “dual” spectral topology whose open sets are precisely the Thomason subsets. The nomenclature comes from the prominent role these “dual-open” sets play in the work of Thomason [Tho97]. In this paper, we will be interested in subsets of spectral spaces that are both Thomason and closed:

Lemma 2.3. *Let \mathcal{K} be a tensor triangulated category and let \mathcal{Z} be a closed subset of $\mathrm{Spc}(\mathcal{K})$. The following are equivalent:*

- (1) \mathcal{Z} is Thomason;
- (2) \mathcal{Z} has quasi-compact complement;
- (3) $\mathcal{Z} = \mathrm{supp}(a)$ for some $a \in \mathcal{K}$.

Proof. We will sketch the proof of (1) implies (2) since (2) clearly implies (1) and [Bal05, Proposition 2.14] gives the equivalence of (2) and (3). For any closed subset $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$, $\mathcal{K}_{\mathcal{Z}} := \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subset \mathcal{Z}\}$ is a thick \otimes -ideal and it is easily checked from the definitions that $\mathrm{Spc}(\mathcal{K}) \setminus \mathcal{Z} \subset \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \supset \mathcal{K}_{\mathcal{Z}}\} =: V(\mathcal{K}_{\mathcal{Z}})$. On the other hand, if \mathcal{Z} is Thomason then one readily checks that the reverse inclusion holds using the equivalence of (2) and (3). Thus, if \mathcal{Z} is Thomason and closed then $\mathrm{Spc}(\mathcal{K}) \setminus \mathcal{Z} = V(\mathcal{K}_{\mathcal{Z}}) \simeq \mathrm{Spc}(\mathcal{K}/\mathcal{K}_{\mathcal{Z}})$ and the spectrum of any tensor triangulated category is quasi-compact (by [Bal05, Corollary 2.15]). \square

Definition 2.4. A *spectral map* between spectral spaces is a continuous map with the property that the preimage of any quasi-compact open subset is again quasi-compact. This is equivalent to being a continuous map that is also continuous with respect to the dual spectral topologies (although one direction of this equivalence is not immediate).

Remark 2.5. Any closed subspace of a spectral space is also spectral, as is the homogeneous spectrum of a graded-commutative graded ring; for the latter, see [BKS07, Proposition 2.5] and [DS, Proposition 2.43]. Our comparison maps will be spectral maps defined on closed subsets of the spectrum of a tensor triangulated category and mapping to the (homogeneous) spectrum of a (graded-)commutative (graded) ring.

Remark 2.6. It is well-known that the Thomason closed subsets of an affine scheme $\mathrm{Spec}(A)$ are those closed sets of the form $V(I)$ for a finitely generated ideal $I \subset A$, and similarly for the homogeneous spectrum of a (graded-)commutative graded ring; cf. Lemma 2.3, (1) \Leftrightarrow (2), above, and [BKS07, Lemma 2.2].

3. BASIC CONSTRUCTIONS

It is now time to introduce the new comparison maps. As mentioned in the introduction, there are actually several different constructions, but they are closely related and the fundamental ideas are exposed in the simplest example. In all cases, there are graded and ungraded versions. The proofs for the graded constructions are essentially the same as for the ungraded ones, but the ideas are more transparent in the ungraded setting. The notion of a “tensor-balanced” endomorphism will play a central role in these constructions.

Definition 3.1. An endomorphism $f : X \rightarrow X$ in a tensor triangulated category is said to be \otimes -balanced if $f \otimes X = X \otimes f$ as an endomorphism of $X \otimes X$.

Remark 3.2. The following lemma was established in [Bal10a, Proposition 2.13] in the case when $f : \mathbb{1} \rightarrow \mathbb{1}$ is an arbitrary endomorphism of the unit and was a crucial technical result used in the construction of the original unit comparison maps. The key to generalizing the result to endomorphisms of an arbitrary object X is to restrict ourselves to \otimes -balanced endomorphisms.

Lemma 3.3. *If $f : X \rightarrow X$ is a \otimes -balanced endomorphism then $f^{\otimes 2} \otimes \mathrm{cone}(f) = 0$.*

Proof. Start with an exact triangle $X \xrightarrow{f} X \xrightarrow{g} \text{cone}(f) \xrightarrow{h} \Sigma X$ and observe that in the following morphism of exact triangles

$$\begin{array}{ccccccc}
X \otimes X & \xrightarrow{X \otimes f} & X \otimes X & \xrightarrow{X \otimes g} & X \otimes \text{cone}(f) & \xrightarrow{X \otimes h} & \Sigma(X \otimes X) \\
f \otimes X \downarrow & & f \otimes X \downarrow & \searrow 0 & \downarrow f \otimes \text{cone}(f) & \searrow 0 & \downarrow \Sigma(f \otimes X) \\
X \otimes X & \xrightarrow{X \otimes f} & X \otimes X & \xrightarrow{X \otimes g} & X \otimes \text{cone}(f) & \xrightarrow{X \otimes h} & \Sigma(X \otimes X)
\end{array}$$

the middle diagonal is zero because $(X \otimes g) \circ (f \otimes X) = (X \otimes g) \circ (X \otimes f) = X \otimes (g \circ f) = 0$ and the rightmost diagonal is zero for similar reasons. This implies that the map $f \otimes \text{cone}(f)$ factors through $X \otimes g$ and $X \otimes h$ and hence $(f \otimes \text{cone}(f))^2 = 0$. It follows that $f^{\otimes 2} \otimes \text{cone}(f) = 0$ by observing that $f^{\otimes 2} = (X \otimes f) \circ (f \otimes X) = X \otimes f^2$. \square

Notation 3.4. Let $E_X := \{f \in [X, X] \mid f \otimes X = X \otimes f\}$ denote the collection of \otimes -balanced endomorphisms of X .

Proposition 3.5. *For each object X in a tensor triangulated category \mathcal{K} , E_X is an inverse-closed subring of the endomorphism ring $[X, X]$. If (0) is a prime in \mathcal{K} , for example if \mathcal{K} is rigid and local, then E_X is a local ring provided that $X \neq 0$.*

Proof. The first statement follows easily from the definitions. On the other hand, suppose that the zero ideal (0) is a prime in \mathcal{K} and that $X \neq 0$. To prove that the non-zero ring E_X is local it suffices to show that the sum of two non-units is again a non-unit. To this end, let $f_1, f_2 \in E_X$ and suppose that $f_1 + f_2$ is a unit. By Lemma 3.3, $f_1^{\otimes 2} \otimes \text{cone}(f_1) = 0$ and $f_2^{\otimes 2} \otimes \text{cone}(f_2) = 0$. It follows that $(f_1 + f_2)^{\otimes n} \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) = 0$ for $n \geq 3$ by expanding $(f_1 + f_2)^{\otimes n}$ using bilinearity of the \otimes -product and applying the symmetry. But the unit $f_1 + f_2$ is a categorical isomorphism and hence any \otimes -power $(f_1 + f_2)^{\otimes n}$ is also an isomorphism. It follows that $X^{\otimes n} \otimes \text{cone}(f_1) \otimes \text{cone}(f_2) = 0$ for $n \geq 3$ and hence that $\text{cone}(f_1) = 0$ or $\text{cone}(f_2) = 0$ since (0) is prime and $X \neq 0$ by assumption. In other words, f_1 or f_2 is an isomorphism (and hence a unit in E_X). \square

Lemma 3.6. *If $F : \mathcal{K} \rightarrow \mathcal{L}$ is a morphism of tensor triangulated categories then the induced ring homomorphism $[X, X]_{\mathcal{K}} \rightarrow [FX, FX]_{\mathcal{L}}$ restricts to a ring homomorphism $E_{\mathcal{K}, X} \rightarrow E_{\mathcal{L}, FX}$.*

Proof. This follows from the fact that $F : \mathcal{K} \rightarrow \mathcal{L}$ is a strong \otimes -functor. \square

Remark 3.7. These results reveal the crucial properties that are secured by restricting ourselves to \otimes -balanced endomorphisms: they provide us with rings of endomorphisms that are local when the category is local, behave well with respect to tensor triangular functors, and have the property that the units are the elements that are categorical isomorphisms. However, these rings are not necessarily commutative.

Theorem 3.8. *Let \mathcal{K} be a tensor triangulated category and let X be an object in \mathcal{K} . For any commutative ring A and ring homomorphism $\alpha : A \rightarrow E_X$ there is an inclusion-reversing, spectral map*

$$\rho_{X,A} : \text{supp}(X) \rightarrow \text{Spec}(A)$$

defined by $\rho_{X,A}(\mathcal{P}) := \{a \in A \mid \text{cone}(\alpha(a)) \notin \mathcal{P}\}$.

Proof. The localization $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{P}$ induces a ring homomorphism $E_{\mathcal{K},X} \rightarrow E_{\mathcal{K}/\mathcal{P},q(X)}$ and since $X \notin \mathcal{P}$ the target ring $E_{\mathcal{K}/\mathcal{P},q(X)}$ is a local ring. For any element $f \in E_{\mathcal{K},X}$ observe that $\text{cone}(f) \notin \mathcal{P}$ iff $q(f)$ is not an isomorphism in \mathcal{K}/\mathcal{P} iff $q(f)$ is a non-unit in the local ring $E_{\mathcal{K}/\mathcal{P},q(X)}$. Since the non-units in a local ring form a two-sided ideal, it follows that the preimage $\{f \in E_{\mathcal{K},X} \mid \text{cone}(f) \notin \mathcal{P}\}$ is a two-sided ideal of $E_{\mathcal{K},X}$. Moreover, this ideal is “prime” in the sense that $\text{cone}(f \cdot g) \notin \mathcal{P}$ implies that $\text{cone}(f) \notin \mathcal{P}$ or $\text{cone}(g) \notin \mathcal{P}$ (by an application of the octahedral axiom). In any case, this “prime” ideal of the non-commutative ring $E_{\mathcal{K},X}$ pulls back via α to a genuine prime ideal $\rho_{X,A}(\mathcal{P})$ of the commutative ring A . This establishes that the map $\rho_{X,A}$ is well-defined and it is clear from the definition that it is inclusion-reversing.

An arbitrary closed set for the Zariski topology on $\text{Spec}(A)$ is of the form $V(E) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supset E\}$ for some subset $E \subset A$. One readily checks that $\rho_{X,A}^{-1}(V(E)) = \bigcap_{a \in E} \text{supp}(\text{cone}(\alpha(a)))$ and we conclude that $\rho_{X,A}$ is continuous. Moreover, if $V(E)$ has quasi-compact complement then $V(E) = V(a_1, \dots, a_n)$ for some finite collection $a_1, \dots, a_n \in A$ and the preimage $\rho_{X,A}^{-1}(V(a_1, \dots, a_n)) = \bigcap_{i=1}^n \text{supp}(\text{cone}(\alpha(a_i))) = \text{supp}(\text{cone}(\alpha(a_1)) \otimes \dots \otimes \text{cone}(\alpha(a_n)))$ also has quasi-compact complement by Lemma 2.3. \square

All the results above have corresponding graded analogues. For a graded endomorphism $f : \Sigma^k X \rightarrow X$ we abuse notation and write $f \otimes X = X \otimes f$ when we really mean that the following diagram commutes:

$$\begin{array}{ccc} \Sigma^k(X \otimes X) & \simeq & \Sigma^k X \otimes X \\ \wr & & \downarrow f \otimes X \\ X \otimes \Sigma^k X & \xrightarrow{X \otimes f} & X \otimes X \end{array} \quad (3.9)$$

Proposition 3.10. *A graded subring E_X^\bullet of the graded endomorphism ring $[X, X]_\bullet$ is defined by setting $E_X^i := \{f \in [X, X]_i \mid f \otimes X = X \otimes f\}$. It has the property that a homogeneous element is a unit in E_X^\bullet iff it is a unit in $[X, X]_\bullet$ iff it is a categorical isomorphism. Moreover, if (0) is a prime in \mathcal{K} , for example if \mathcal{K} is rigid and local, then E_X^\bullet is gr-local provided that $X \neq 0$.*

Proof. The proof is similar to the ungraded version, one just needs to take the relevant suspension isomorphisms into account. In particular, the result of Lemma 3.3 holds for a graded endomorphism $f : \Sigma^k X \rightarrow X$ satisfying $f \otimes X = X \otimes f$. On the other hand, one could save time and conclude that E_X^\bullet is gr-local simply by invoking the fact that a \mathbb{Z} -graded ring E_X^\bullet is gr-local iff $E_X^0 = E_X$ is local (see [Li12, Theorem 2.5]). \square

Lemma 3.11. *If $F : \mathcal{K} \rightarrow \mathcal{L}$ is a morphism of tensor triangulated categories then the induced graded ring homomorphism $[X, X]_{\mathcal{K},\bullet} \rightarrow [FX, FX]_{\mathcal{L},\bullet}$ restricts to a graded ring homomorphism $E_{\mathcal{K},X}^\bullet \rightarrow E_{\mathcal{L},FX}^\bullet$.*

Proof. This involves verifying that a diagram commutes using the monoidal nature of the functor. The subtle point is that because of the suspension isomorphisms involved in (3.9) one must utilize the compatibility axiom (1.3) for morphisms of tensor triangulated categories. \square

Theorem 3.12. *Let \mathcal{K} be a tensor triangulated category and let X be an object in \mathcal{K} . For any (graded-)commutative graded ring A^\bullet and graded ring homomorphism $\alpha : A^\bullet \rightarrow E_X^\bullet$ there is an inclusion-reversing, spectral map*

$$\rho_{X,A^\bullet}^\bullet : \text{supp}(X) \rightarrow \text{Spec}^h(A^\bullet)$$

defined by $\rho_{X,A^\bullet}^\bullet(\mathcal{P}) := \{a \in A^i \mid \text{cone}(\alpha(a)) \notin \mathcal{P}\}_{i \in \mathbb{Z}}$.

Proof. The localization functor $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{P}$ induces a graded ring homomorphism $E_{\mathcal{K},X}^\bullet \rightarrow E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$ and since $X \notin \mathcal{P}$ the target ring $E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$ is gr-local. The homogeneous non-units in $E_{\mathcal{K}/\mathcal{P},q(X)}^\bullet$ therefore form a two-sided ideal; moreover, an application of the octahedral axiom shows that this ideal is prime in the sense that the product of two homogeneous units is again a unit. The pullback of this ideal to A^\bullet is a genuine homogeneous prime ideal of A^\bullet and this is exactly what $\rho_{X,A^\bullet}^\bullet(\mathcal{P})$ is defined to be. This shows that the map $\rho_{X,A^\bullet}^\bullet$ is well-defined and it is clear from the definition that it is inclusion-reversing. Showing that it is spectral involves an argument similar to the one given in the proof of Theorem 3.8. \square

Remark 3.13. It is clear from the definitions that there is a commutative diagram

$$\begin{array}{ccc} \text{supp}(X) & \xrightarrow{\rho_{X,A^\bullet}^\bullet} & \text{Spec}^h(A^\bullet) \\ & \searrow \rho_{X,A^0} & \downarrow (-)^0 \\ & & \text{Spec}(A^0) \end{array}$$

where $(-)^0$ is the surjective spectral map sending $\mathfrak{p} \in \text{Spec}^h(A^\bullet)$ to $\mathfrak{p}^0 = \mathfrak{p} \cap A^0$. The surjectivity of this map is explained in [Bal10a, Remark 5.5].

Example 3.14. Any (graded-)commutative graded subring of E_X^\bullet yields an associated comparison map. Obvious examples include the graded-center of E_X^\bullet and $\{f \in \text{Center}[X, X]_\bullet \mid f \otimes X = X \otimes f\}$. A more exotic example is given by

$$\{f \in \text{Center}[X, X]_\bullet \mid f \otimes X \in \text{Center}[X^{\otimes 2}, X^{\otimes 2}]_\bullet\}. \quad (3.15)$$

For this third example, note that if $f \otimes X \in \text{Center}[X^{\otimes 2}, X^{\otimes 2}]_\bullet$ then it follows from the fact that the symmetry $\tau : X \otimes X \xrightarrow{\sim} X \otimes X$ is in $[X^{\otimes 2}, X^{\otimes 2}]$ that $f \in E_X^\bullet$; so (3.15) does indeed give a graded subring of E_X^\bullet .

Example 3.16. If $X = \mathbb{1}$ then the condition $f \otimes X = X \otimes f$ holds for any graded endomorphism and it is well-known that the ring $E_{\mathbb{1}}^\bullet = [\mathbb{1}, \mathbb{1}]_\bullet$ is graded-commutative. The map $\rho_{\mathbb{1},[\mathbb{1},\mathbb{1}]_\bullet}^\bullet$ is the original graded comparison map from [Bal10a].

Example 3.17. Recall the notion of the graded-center $Z^\bullet(\mathcal{T})$ of a triangulated category \mathcal{T} (see [KY11], for example). For a tensor triangulated category \mathcal{T} one can define a graded-commutative graded subring of $Z^\bullet(\mathcal{T})$ by setting

$$Z_{\otimes}^i(\mathcal{T}) := \{\alpha \in Z^i(\mathcal{T}) \mid X \otimes \alpha_Y = \alpha_{X \otimes Y} \text{ for every } X, Y \in \mathcal{T}\}$$

for each $i \in \mathbb{Z}$. Observe that any $\alpha \in Z_{\otimes}^i(\mathcal{T})$ is completely determined by $\alpha_{\mathbb{1}}$ and there is an obvious isomorphism $Z_{\otimes}^\bullet(\mathcal{T}) \xrightarrow{\sim} [\mathbb{1}, \mathbb{1}]_\bullet$. However, the definition makes sense for any thick \otimes -ideal $\mathcal{I} \subseteq \mathcal{T}$ and $Z_{\otimes}^\bullet(\mathcal{I})$ is not obviously so trivial for $\mathcal{I} \subsetneq \mathcal{T}$. For any object $X \in \mathcal{I}$ there is a graded ring homomorphism $Z_{\otimes}^\bullet(\mathcal{I}) \rightarrow E_X^\bullet$ given by $\alpha \mapsto \alpha_X$ and so we obtain a map $\text{supp}(X) \rightarrow \text{Spec}^h(Z_{\otimes}^\bullet(\mathcal{I}))$. Explicitly, it maps a prime \mathcal{P} to $\{\alpha \in Z_{\otimes}^\bullet(\mathcal{I}) \mid \text{cone}(\alpha_X) \notin \mathcal{P}\}$. However, for a fixed prime \mathcal{P} and a fixed

$\alpha \in Z_{\otimes}^{\bullet}(\mathcal{I})$ the set $\{Y \in \mathcal{I} \mid \text{cone}(\alpha_Y) \in \mathcal{P}\}$ is readily checked to be a thick \otimes -ideal. It follows that if X generates \mathcal{I} as a thick \otimes -ideal then $\{\alpha \in Z_{\otimes}^{\bullet}(\mathcal{I}) \mid \text{cone}(\alpha_X) \notin \mathcal{P}\}$ is the same as $\{\alpha \in Z_{\otimes}^{\bullet}(\mathcal{I}) \mid \text{cone}(\alpha_Y) \notin \mathcal{P} \text{ for some } Y \in \mathcal{I}\}$. In other words, if \mathcal{I} is generated as a thick \otimes -ideal by a single object (equivalently, by a finite number of objects) then every generator gives the exact same comparison map. In conclusion, every finitely generated thick \otimes -ideal \mathcal{I} has an associated (generator-independent) comparison map $\text{supp}(\mathcal{I}) \rightarrow \text{Spec}^h(Z_{\otimes}^{\bullet}(\mathcal{I}))$ which sends a prime \mathcal{P} to $\{\alpha \in Z_{\otimes}^{\bullet}(\mathcal{I}) \mid \text{cone}(\alpha_Y) \notin \mathcal{P} \text{ for some } Y \in \mathcal{I}\}$.

Remark 3.18. In the proof of Theorem 3.8, we saw how to associate a “prime” ideal of the non-commutative ring E_X to any prime $\mathcal{P} \in \text{supp}(X)$ which was then pulled back to a genuine prime ideal of a commutative ring A via a map $A \rightarrow E_X$. A suitable theory of spectra for non-commutative rings might allow us to work directly with the ring E_X but this avenue has not been pursued. In any case, taking commutative rings mapping into E_X is a flexible approach which provides for some interesting examples not obviously tied to the ring E_X (e.g., Example 3.17 above). On the other hand, although the maps $\rho_{X,A}$ are useful for some purposes, they will not typically be natural with respect to tensor triangular functors. The problem is that although the construction of the ring E_X is functorial (recall Lemma 3.6), the construction of various commutative rings A mapping into E_X will typically not be. For example, the center of E_X is not a functorial construction, nor is the graded-center of a triangulated category. In the next section, we will replace E_X by a functorial commutative ring R_X and obtain a comparison map $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$ which is natural with respect to tensor triangular functors. In fact, the construction of ρ_X will be a special case of a much more general construction, which will provide us with additional examples of natural comparison maps.

4. NATURAL CONSTRUCTIONS

Let Φ be a non-empty set of objects in a tensor triangulated category \mathcal{K} that is closed under the \otimes -product ($a, b \in \Phi \Rightarrow a \otimes b \in \Phi$). For any object $X \in \mathcal{K}$ recall that E_X denotes the ring of \otimes -balanced endomorphisms of X .

Lemma 4.1. *Suppose $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are two endomorphisms with $g \in E_Y$. If $X \otimes f \otimes g = f \otimes X \otimes g$ then $f \otimes g \in E_{X \otimes Y}$.*

Proof. The commutativity of the diagram

$$\begin{array}{ccc}
 X \otimes Y \otimes X \otimes Y & \xrightarrow{X \otimes Y \otimes f \otimes g} & X \otimes Y \otimes X \otimes Y \\
 \downarrow \tau \otimes X \otimes Y & & \downarrow \tau \otimes X \otimes Y \\
 \text{id} \left(Y \otimes X \otimes X \otimes Y \right) & \xrightarrow[\begin{smallmatrix} Y \otimes X \otimes f \otimes g \\ Y \otimes f \otimes X \otimes g \end{smallmatrix}]{} & Y \otimes X \otimes X \otimes Y \text{id} \\
 \downarrow \tau \otimes X \otimes Y & & \downarrow \tau \otimes X \otimes Y \\
 X \otimes Y \otimes X \otimes Y & \xrightarrow{f \otimes Y \otimes X \otimes g} & X \otimes Y \otimes X \otimes Y \\
 \downarrow X \otimes \tau \otimes Y & & \downarrow X \otimes \tau \otimes Y \\
 \text{id} \left(X \otimes X \otimes Y \otimes Y \right) & \xrightarrow[\begin{smallmatrix} f \otimes X \otimes Y \otimes g \\ f \otimes X \otimes g \otimes Y \end{smallmatrix}]{} & X \otimes X \otimes Y \otimes Y \text{id} \\
 \downarrow X \otimes \tau \otimes Y & & \downarrow X \otimes \tau \otimes Y \\
 X \otimes Y \otimes X \otimes Y & \xrightarrow{f \otimes g \otimes X \otimes Y} & X \otimes Y \otimes X \otimes Y
 \end{array}$$

verifies that $f \otimes g \in E_{X \otimes Y}$. \square

Corollary 4.2. *For any pair of objects $X, Y \in \mathcal{K}$ the functors $- \otimes Y$ and $X \otimes -$ induce ring homomorphisms $E_X \rightarrow E_{X \otimes Y}$ and $E_Y \rightarrow E_{X \otimes Y}$.*

Definition 4.3. Define R_Φ to be the set $\{(X, f) : X \in \Phi, f \in E_X\} / \sim$ where \sim is the smallest equivalence relation such that $(X, f) \sim (a \otimes X, a \otimes f)$ and $(X, f) \sim (X \otimes a, f \otimes a)$ for every $a \in \Phi$.

Notation 4.4. A subscript f_X will indicate that f is an endomorphism of X and $[f_X]$ will denote the image of (X, f) in R_Φ .

Lemma 4.5. *For any isomorphism $\alpha : X \xrightarrow{\sim} Y$ in \mathcal{K} , the isomorphism of rings $[X, X] \xrightarrow{\sim} [Y, Y]$ given by $f \mapsto \alpha \circ f \circ \alpha^{-1}$ restricts to give an isomorphism $\alpha_* : E_X \xrightarrow{\sim} E_Y$. If $f \in E_X$ then $f \otimes Y = X \otimes \alpha_*(f)$ as an endomorphism of $X \otimes Y$.*

Proof. This is routine from the definitions. \square

Notation 4.6. For two endomorphisms f_X and g_Y the notation $f_X \simeq g_Y$ will indicate that there exists an isomorphism $\alpha : X \xrightarrow{\sim} Y$ such that $\alpha_*(f_X) = g_Y$.

Remark 4.7. The above lemma implies that in R_Φ endomorphisms of X are identified with endomorphisms of Y via all isomorphisms $X \xrightarrow{\sim} Y$. In particular, an endomorphism f_X is identified with the “twisted” version $\sigma \circ f \circ \sigma^{-1}$ for every automorphism $\sigma : X \rightarrow X$. Because of these identifications, an essentially equivalent approach to the construction of R_Φ could be obtained by taking Φ to be a set of isomorphism classes of objects closed under the \otimes -product. However, such an approach would obscure the fact that these identifications up to isomorphism are forced by the innocuous identifications $f \sim a \otimes f$ and $f \sim f \otimes a$.

Lemma 4.8. *Two endomorphisms f_X and g_Y are equivalent in R_Φ if and only if there exist objects $a, b \in \Phi$ such that $a \otimes f_X \simeq b \otimes g_Y$ if and only if there exists an object $c \in \Phi$ such that $f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$.*

Proof. To show that the equivalence relation defined by the third condition (existence of $c \in \Phi$ such that $f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$) is stronger than the equivalence relation defining R_Φ we need to show that $f \in E_X$ is identified with $a \otimes f$ and $f \otimes a$ for any $a \in \Phi$. Indeed, the fact that f_X is \otimes -balanced implies that $f \otimes c \otimes a \otimes X = X \otimes c \otimes a \otimes f$ and $f \otimes c \otimes X \otimes a = X \otimes c \otimes f \otimes a$ for any object $c \in \mathcal{K}$ (cf. the proof of Lemma 4.1). On the other hand, $f_X \otimes c \otimes Y = X \otimes c \otimes g_Y$ clearly implies that $[f_X] = [g_Y]$ in R_Φ . Similarly, the second condition defines an equivalence relation which is evidently stronger than the one defining R_Φ . On the other hand, if $a \otimes f_X \simeq b \otimes g_Y$ then by definition there exists an isomorphism $\alpha : a \otimes X \xrightarrow{\sim} b \otimes Y$ such that $b \otimes g_Y = \alpha_*(a \otimes f_X)$. Lemma 4.5 then implies that $a \otimes f_X \otimes b \otimes Y = a \otimes X \otimes b \otimes g_Y$ so that $[f_X] = [g_Y]$ in R_Φ . \square

Proposition 4.9. *The set R_Φ is a commutative ring with addition and multiplication defined by $[f_X] + [g_Y] := [f_X \otimes Y + X \otimes g_Y]$ and $[f_X] \cdot [g_Y] := [(f_X \otimes Y) \circ (X \otimes g_Y)]$. It is non-zero provided that Φ does not contain a zero object. If (0) is a prime in \mathcal{K} , for example if \mathcal{K} is rigid and local, then R_Φ is a local ring provided that it is non-zero.*

Proof. Armed with Lemma 4.8, it is a long but straightforward exercise to establish that addition and multiplication are well-defined and endow R_Φ with a ring

structure. There are several ways to see that this ring structure is commutative. For example, the fact that f_X is \otimes -balanced implies that $f \otimes g \otimes X = X \otimes g \otimes f$ and so $[f] \cdot [g] = [f \otimes g] = [f \otimes g \otimes X] = [X \otimes g \otimes f] = [g \otimes f] = [g] \cdot [f]$. One readily checks that $[f] = 0$ iff $a \otimes f = 0$ for some $a \in \Phi$ and that $[f]$ is a unit iff $a \otimes f$ is an isomorphism for some $a \in \Phi$. The proof that R_Φ is local when (0) is prime closely mirrors the proof that E_X is local (Proposition 3.5). Indeed, if $[f_X] + [g_Y]$ is a unit in R_Φ then $a \otimes (f \otimes Y + X \otimes g)$ is an isomorphism for some $a \in \Phi$. It follows that $a^{\otimes n} \otimes (f \otimes Y + X \otimes g)^{\otimes n} \otimes \text{cone}(f) \otimes \text{cone}(g)$ is both zero and an isomorphism for $n \geq 3$. This implies that $a^{\otimes n} \otimes (X \otimes Y)^{\otimes n} \otimes \text{cone}(f) \otimes \text{cone}(g) = 0$ for $n \geq 3$ and since (0) is prime and $X, Y, a \in \Phi$ are non-zero we conclude that f or g is an isomorphism (and hence $[f]$ or $[g]$ is a unit in R_Φ). \square

Remark 4.10. The ring R_Φ is the colimit of a diagram of rings consisting of E_X for each $X \in \Phi$ with maps generated by $a \otimes - : E_X \rightarrow E_{a \otimes X}$ and $- \otimes a : E_X \rightarrow E_{X \otimes a}$. Although the index category on which this diagram is defined is not technically a filtered category (because there are parallel arrows that are not coequalized in the category), $\text{colim}_{X \in \Phi} E_X$ is still a filtered colimit in the more general sense of [Sta, Chapter 4, Section 17] which is sufficient for the colimit to be created in the category of sets. Rather than define R_Φ to be this colimit (which would necessitate a longer discussion of these technicalities) we have opted for the concrete description given above.

Proposition 4.11. *Let $F : \mathcal{K} \rightarrow \mathcal{L}$ be a morphism of tensor triangulated categories. Suppose $\Phi \subset \mathcal{K}$ and $\Psi \subset \mathcal{L}$ are non-empty \otimes -multiplicative subsets such that $F(\Phi) \subset \Psi$. Then $[f] \mapsto [F(f)]$ defines a ring homomorphism $R_{\mathcal{K}, \Phi} \rightarrow R_{\mathcal{L}, \Psi}$.*

Proof. This is a routine verification using Lemma 4.8 and the fact that $F : \mathcal{K} \rightarrow \mathcal{L}$ is a strong \otimes -functor. \square

Theorem 4.12. *Let \mathcal{K} be a tensor triangulated category and let $\Phi \subset \mathcal{K}$ be a non-empty set of objects closed under the \otimes -product. Let $\mathcal{Z}_\Phi := \bigcap_{X \in \Phi} \text{supp}(X)$. There is an inclusion-reversing, spectral map*

$$\rho_\Phi : \mathcal{Z}_\Phi \rightarrow \text{Spec}(R_\Phi)$$

defined by $\mathcal{P} \mapsto \{[f] \in R_\Phi \mid \text{cone}(f) \notin \mathcal{P}\}$.

Proof. The first point to make is that for $\mathcal{P} \in \mathcal{Z}_\Phi$ the condition $\text{cone}(f) \notin \mathcal{P}$ does not depend on the choice of representative of $[f] \in R_\Phi$. Indeed, if $[f] = [g]$ then $a \otimes f \simeq b \otimes g$ for some $a, b \in \Phi$, so $\text{cone}(a \otimes f) \simeq \text{cone}(b \otimes g)$. If $\mathcal{P} \in \mathcal{Z}_\Phi$ then $\mathcal{P} \in \text{supp}(\text{cone}(f))$ iff $\mathcal{P} \in \text{supp}(a) \cap \text{supp}(\text{cone}(f)) = \text{supp}(b) \cap \text{supp}(\text{cone}(g))$ iff $\mathcal{P} \in \text{supp}(\text{cone}(g))$. The second point to make is that for any prime $\mathcal{P} \in \text{Spc}(\mathcal{K})$ the quotient functor $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{P}$ induces a ring homomorphism $R_{\mathcal{K}, \Phi} \rightarrow R_{\mathcal{K}/\mathcal{P}, q(\Phi)}$ whose target ring is local provided that $0 \notin q(\Phi)$; in other words, provided that $\mathcal{P} \cap \Phi = \emptyset$ which is equivalent to saying that $\mathcal{P} \in \mathcal{Z}_\Phi$. With these facts in mind the proof is similar to the proof of Theorem 3.8. For an element $[f] \in R_{\mathcal{K}, \Phi}$, $[q(f)]$ is a unit in $R_{\mathcal{K}/\mathcal{P}, q(\Phi)}$ iff there exists $a \in \Phi$ such that $q(a \otimes f)$ is an isomorphism in \mathcal{K}/\mathcal{P} iff there exists $a \in \Phi$ such that $a \otimes \text{cone}(f) \in \mathcal{P}$ iff $\text{cone}(f) \in \mathcal{P}$. Thus $\rho_\Phi(\mathcal{P})$ is the pullback of the collection of non-units in $R_{\mathcal{K}/\mathcal{P}, q(\Phi)}$ and since the non-units in a local ring form a (two-sided) ideal, this establishes that $\rho_\Phi(\mathcal{P})$ is an ideal. It is prime since $[f_X] \cdot [g_Y] \in \rho_\Phi(\mathcal{P})$ implies that $\mathcal{P} \in \text{supp}(\text{cone}(f \otimes g)) \subset \text{supp}(\text{cone}(f \otimes Y)) \cup \text{supp}(\text{cone}(X \otimes g)) \subset \text{supp}(\text{cone}(f)) \cup \text{supp}(\text{cone}(g))$. That ρ_Φ

is a spectral map is established by similar modifications to the argument given in the proof of Theorem 3.8. \square

Example 4.13. For any object $X \in \mathcal{K}$, taking $\Phi := \{X^{\otimes n} \mid n \geq 1\}$ provides a comparison map $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$. These are the “object” comparison maps mentioned in the introduction.

Example 4.14. For any closed set $\mathcal{Z} \subset \text{Spc}(\mathcal{K})$, taking $\Phi := \{a \in \mathcal{K} \mid \text{supp}(a) \supset \mathcal{Z}\}$ gives a comparison map $\rho_{\mathcal{Z}} : \mathcal{Z} \rightarrow \text{Spec}(R_{\mathcal{Z}})$. These are the “closed set” comparison maps mentioned in the introduction. Note that $\mathcal{Z}_{\Phi} = \mathcal{Z}$ because $\{\text{supp}(a) : a \in \mathcal{K}\}$ forms a basis of closed sets for the topology on $\text{Spc}(\mathcal{K})$. Also note that if the category \mathcal{K} is not small then there is the unfortunate detail that Φ might not be a set; however, we do not need to worry about this technicality because of Remark 4.7.

Example 4.15. For any Thomason closed set $\mathcal{Z} \subset \text{Spc}(\mathcal{K})$, taking $\Phi := \{a \in \mathcal{K} \mid \text{supp}(a) = \mathcal{Z}\}$ provides another comparison map defined on \mathcal{Z} . However, the ring R_{Φ} is canonically isomorphic to the one in Example 4.14 and under this identification the two comparison maps coincide. In other words, when \mathcal{Z} is Thomason we can take the target ring of the “closed set” comparison map $\rho_{\mathcal{Z}} : \mathcal{Z} \rightarrow \text{Spec}(R_{\mathcal{Z}})$ to be defined using only those objects X for which $\text{supp}(X) = \mathcal{Z}$.

Example 4.16. Another candidate to consider would be $\Phi := \{a \in \mathcal{K} \mid \text{supp}(a) \subset \mathcal{Z}\}$ but in this case $\mathcal{Z}_{\Phi} = \emptyset$ and $R_{\Phi} = 0$. Indeed, it is easily checked that $R_{\Phi} = 0$ iff $0 \in \Phi$ iff Φ contains an object that is \otimes -nilpotent iff $\mathcal{Z}_{\Phi} = \emptyset$.

Remark 4.17. There are other examples that could be considered, such as the collection of \otimes -invertible objects, or the collection of objects that are isomorphic to a direct sum of suspensions of $\mathbf{1}$.

Remark 4.18. For any commutative ring A and ring homomorphism $A \rightarrow R_{\Phi}$ one obtains an inclusion-reversing, spectral map $\mathcal{Z}_{\Phi} \rightarrow \text{Spec}(A)$ by composing ρ_{Φ} with the induced map $\text{Spec}(R_{\Phi}) \rightarrow \text{Spec}(A)$. For example, the comparison maps $\rho_{X,A} : \text{supp}(X) \rightarrow \text{Spec}(A)$ defined in Section 3 are recovered from $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$ by composing $A \rightarrow E_X$ with the canonical map $E_X \rightarrow R_X$. In fact, the ring R_{Φ} can be regarded as a colimit of all the commutative rings A mapping into the rings E_X for $X \in \Phi$. More precisely, consider triples (A, α, X) where A is a commutative ring, $\alpha : A \rightarrow E_X$ is a ring homomorphism, and X is an object of Φ . Define a morphism $(A, \alpha, X) \rightarrow (A', \alpha', X')$ to be a morphism $u : A \rightarrow A'$ such that

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & E_X \\ u \downarrow & & \searrow \\ A' & \xrightarrow{\alpha'} & E_{X'} \end{array} \quad \begin{array}{c} \\ \\ \nearrow \\ \end{array} \quad \begin{array}{c} \\ \\ E_{X \otimes a \otimes X'} \end{array}$$

commutes for some object $a \in \Phi$. (If the object a were not included then one would run into difficulties composing such morphisms.) There is an obvious functor $(A, \alpha, X) \mapsto A$ from the category of triples to the category of rings and it is easy to check that the maps $A \xrightarrow{\alpha} E_X \rightarrow R_{\Phi}$ induce a morphism $\text{colim}_{(A, \alpha, X)} A \rightarrow R_{\Phi}$. Let us briefly sketch the proof that this is an isomorphism. For surjectivity, one observes that if $[f_X] \in R_{\Phi}$ then the subring $\mathbb{Z}[f_X] \subset E_X$ generated by f_X is commutative and provides a triple $(\mathbb{Z}[f_X], i, X)$. On the other hand, to establish injectivity one first proves that the category of triples is filtered. It follows that every

element of $\text{colim}_{(A,\alpha,X)} A$ is the image of an element $x \in A$ under the canonical map $A \rightarrow \text{colim}_{(A,\alpha,X)} A$ associated with a triple (A, α, X) . If the element in $\text{colim}_{(A,\alpha,X)} A$ goes to zero in R_Φ then $[\alpha(x)] = 0$ in R_Φ and so there is some $a \in \Phi$ such that $a \otimes \alpha(x) = 0$. If β is the composite $A \xrightarrow{\alpha} E_X \rightarrow E_{a \otimes X}$ then the induced map $\bar{\beta} : A/\ker \beta \rightarrow E_{a \otimes X}$ provides a triple $(A/\ker \beta, \bar{\beta}, a \otimes X)$ while the quotient map $\pi : A \rightarrow A/\ker \beta$ defines a map of triples $(A, \alpha, X) \rightarrow (A/\ker \beta, \bar{\beta}, a \otimes X)$. Since the canonical map $A \rightarrow \text{colim}_{(A,\alpha,X)} A$ for the triple (A, α, X) factors through the above map of triples, we conclude that the image of x in $\text{colim}_{(A,\alpha,X)} A$ is zero. From this perspective, the maps defined in Section 3 are obtained from ρ_Φ by pulling back via the canonical map $A \rightarrow \text{colim}_{(A,\alpha,X)} A \simeq R_\Phi$ associated to a triple (A, α, X) .

Proposition 4.19. *Let $F : (\mathcal{K}, \Phi) \rightarrow (\mathcal{L}, \Psi)$ be a morphism of tensor triangulated categories $\mathcal{K} \rightarrow \mathcal{L}$ such that $F(\Phi) \subset \Psi$. Then there is a commutative diagram*

$$\begin{array}{ccc} \text{Spc}(\mathcal{L}) \supset \mathcal{Z}_{\mathcal{L},\Psi} & \xrightarrow{\rho_{\mathcal{L},\Psi}} & \text{Spec}(R_{\mathcal{L},\Psi}) \\ \downarrow & & \downarrow \\ \text{Spc}(\mathcal{K}) \supset \mathcal{Z}_{\mathcal{K},\Phi} & \xrightarrow{\rho_{\mathcal{K},\Phi}} & \text{Spec}(R_{\mathcal{K},\Phi}) \end{array} \quad (4.20)$$

in the category of spectral spaces.

Proof. A ring homomorphism $R_{\mathcal{K},\Phi} \rightarrow R_{\mathcal{L},\Psi}$ is provided by Proposition 4.11 and the rest is a routine recollection of the relevant definitions. \square

Example 4.21. If $\mathcal{Z}_1 \subset \mathcal{Z}_2$ is an inclusion of closed subsets then there is a ring homomorphism $R_{\mathcal{Z}_2} \rightarrow R_{\mathcal{Z}_1}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_2 & \xrightarrow{\rho_{\mathcal{Z}_2}} & \text{Spec}(R_{\mathcal{Z}_2}) \\ \cup & & \uparrow \\ \mathcal{Z}_1 & \xrightarrow{\rho_{\mathcal{Z}_1}} & \text{Spec}(R_{\mathcal{Z}_1}). \end{array}$$

Example 4.22. If \mathcal{Z} is a Thomason closed subset then for any object X with $\text{supp}(X) = \mathcal{Z}$ there is a ring homomorphism $R_X \rightarrow R_{\mathcal{Z}}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\rho_{\mathcal{Z}}} & \text{Spec}(R_{\mathcal{Z}}) \\ & \searrow \rho_X & \downarrow \\ & & \text{Spec}(R_X). \end{array}$$

It is worth explicitly stating the naturality in the case of the object and closed set comparison maps:

Proposition 4.23. *If $F : \mathcal{K} \rightarrow \mathcal{L}$ is a morphism of tensor triangulated categories and $X \in \mathcal{K}$ then there is a commutative diagram*

$$\begin{array}{ccc} \text{Spc}(\mathcal{L}) \supset \text{supp}_{\mathcal{L}}(FX) & \xrightarrow{\rho_{\mathcal{L},FX}} & \text{Spec}(R_{\mathcal{L},FX}) \\ \downarrow & & \downarrow \\ \text{Spc}(\mathcal{K}) \supset \text{supp}_{\mathcal{K}}(X) & \xrightarrow{\rho_{\mathcal{K},X}} & \text{Spec}(R_{\mathcal{K},X}) \end{array}$$

in the category of spectral spaces, where the left square is cartesian.

Proposition 4.24. *If $F : \mathcal{K} \rightarrow \mathcal{L}$ is a morphism of small tensor triangulated categories and $\mathcal{Z} \subset \mathrm{Spc}(\mathcal{K})$ is a closed subset then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spc}(\mathcal{L}) \supset \phi^{-1}(\mathcal{Z}) & \xrightarrow{\rho_{\mathcal{L}, \phi^{-1}(\mathcal{Z})}} & \mathrm{Spec}(R_{\mathcal{L}, \phi^{-1}(\mathcal{Z})}) \\ \phi \downarrow & & \downarrow \\ \mathrm{Spc}(\mathcal{K}) \supset \mathcal{Z} & \xrightarrow{\rho_{\mathcal{K}, \mathcal{Z}}} & \mathrm{Spec}(R_{\mathcal{K}, \mathcal{Z}}) \end{array}$$

in the category of spectral spaces, where the left square is cartesian.

Remark 4.25. Considering $\mathrm{supp}_{\mathcal{K}}(X)$ and $\mathrm{Spec}(R_{\mathcal{K}, X})$ as contravariant functors from the category of essentially small tensor triangulated categories with chosen object to the category of spectral spaces, the object comparison maps $\rho_{\mathcal{K}, X}$ can be regarded as a natural transformation $\mathrm{supp}_{\mathcal{K}}(X) \rightarrow \mathrm{Spec}(R_{\mathcal{K}, X})$. Similarly, there is a contravariant “forgetful” functor $(\mathcal{K}, \mathcal{Z}) \mapsto \mathcal{Z}$ from the category of small tensor triangulated categories with chosen closed subset of their spectrum to the category of spectral spaces, and the closed set comparison maps $\rho_{\mathcal{K}, \mathcal{Z}}$ form a natural transformation from this functor to the functor $(\mathcal{K}, \mathcal{Z}) \mapsto \mathrm{Spec}(R_{\mathcal{K}, \mathcal{Z}})$. Finally, $\rho_{\mathcal{K}, \Phi}$ can be regarded as a natural transformation from $(\mathcal{K}, \Phi) \mapsto \mathcal{Z}_{\Phi}$ to $(\mathcal{K}, \Phi) \mapsto \mathrm{Spec}(R_{\mathcal{K}, \Phi})$.

Remark 4.26. It is straightforward to develop the graded version of these constructions. One checks that the graded analogue of Corollary 4.2 holds and then defines R_{Φ}^{\bullet} to be the colimit of the diagram of graded rings generated by the maps $E_X^{\bullet} \rightarrow E_{X \otimes Y}^{\bullet}$ and $E_Y^{\bullet} \rightarrow E_{X \otimes Y}^{\bullet}$. One checks that this is a filtered colimit (in the weak sense—see Remark 4.10) and it is easily determined how filtered colimits of graded rings are constructed. To be clear, the abelian group R_{Φ}^i is the filtered colimit of abelian groups $\mathrm{colim}_{X \in \Phi} E_X^i$ and thus consists of equivalence classes $[f]$ where $f \in E_X^i$ for some $X \in \Phi$. The product on R_{Φ}^{\bullet} is given by

$$\begin{aligned} \mathrm{colim}_{X \in \Phi} E_X^i \times \mathrm{colim}_{Y \in \Phi} E_Y^j &\longrightarrow \mathrm{colim}_{Z \in \Phi} E_Z^{i+j} \\ ([f_X], [g_Y]) &\longmapsto [(f_X \otimes Y) \cdot (X \otimes g_Y)] \end{aligned}$$

where $(f_X \otimes Y) \cdot (X \otimes g_Y)$ is the graded product in $E_{X \otimes Y}^{\bullet}$. Note that R_{Φ}^0 is exactly the ungraded ring R_{Φ} from Definition 4.3. It is straightforward to show that R_{Φ}^{\bullet} is graded-commutative although one needs to be clear about our abuses of notation concerning the suspension isomorphisms. Ultimately the graded-commutativity comes from the anti-commutativity of diagram (1.2) in the axioms of a tensor triangulated category.

The proof of the following theorem is very similar to the proof of Theorem 3.12 just with the kind of modifications we saw in the proof of Theorem 4.12.

Theorem 4.27. *Let \mathcal{K} be a tensor triangulated category and let $\Phi \subset \mathcal{K}$ be a non-empty set of objects closed under the \otimes -product. There is a graded-commutative graded ring R_{Φ}^{\bullet} and an inclusion-reversing, spectral map*

$$\rho_{\Phi}^{\bullet} : \mathcal{Z}_{\Phi} \rightarrow \mathrm{Spec}^h(R_{\Phi}^{\bullet})$$

defined by $\rho_{\Phi}^{\bullet}(\mathcal{P}) := \{[f] \in R_{\Phi}^i \mid \mathrm{cone}(f) \notin \mathcal{P}\}_{i \in \mathbb{Z}}$. The ring R_{Φ} is precisely R_{Φ}^0 and $\mathfrak{p}^{\bullet} \mapsto \mathfrak{p}^{\bullet} \cap R_{\Phi}^0$ defines a surjective spectral map $\mathrm{Spec}^h(R_{\Phi}^{\bullet}) \rightarrow \mathrm{Spec}(R_{\Phi})$ such that

the following diagram commutes

$$\begin{array}{ccc} \mathcal{Z}_\Phi & \xrightarrow{\rho_\Phi^\bullet} & \mathrm{Spec}^h(R_\Phi^\bullet) \\ & \searrow \rho_\Phi & \downarrow (-)^0 \\ & & \mathrm{Spec}(R_\Phi). \end{array}$$

Remark 4.28. The graded comparison maps have the same kind of naturality properties as the ungraded comparison maps.

5. OBJECT COMPARISON MAPS

In this section we will establish some of the basic features of the natural “object” comparison maps $\rho_X : \mathrm{supp}(X) \rightarrow \mathrm{Spec}(R_X)$ defined in Example 4.13. More specifically, our primary goal is to establish that ρ_X is invariant under some natural operations that can be performed on the object X such as taking duals, or suspensions, or \otimes -powers, etc. Before we begin proving such results, let us remark that for $X = \mathbb{1}$ the canonical map $[\mathbb{1}, \mathbb{1}] = E_{\mathbb{1}} \rightarrow R_{\mathbb{1}}$ is an isomorphism and under this identification $\rho_{\mathbb{1}} : \mathrm{Spc}(\mathcal{K}) \rightarrow \mathrm{Spec}(R_{\mathbb{1}})$ is the original unit comparison map from [Bal10a]; similarly for the graded version. It will also be convenient to recognize that the canonical homomorphisms $E_{X^{\otimes n}} \rightarrow R_X$ induce an isomorphism

$$R_X \simeq \mathrm{colim}(E_X \xrightarrow{X^{\otimes -}} E_{X^{\otimes 2}} \xrightarrow{X^{\otimes -}} E_{X^{\otimes 3}} \xrightarrow{X^{\otimes -}} E_{X^{\otimes 4}} \xrightarrow{X^{\otimes -}} \dots) \quad (5.1)$$

and we will often tacitly make the identification $R_X = \mathrm{colim}_{n \geq 1} E_{X^{\otimes n}}$.

Lemma 5.2. *If $f \in E_X$ then $\langle \mathrm{cone}(f) \rangle \subset \{a \in \mathcal{K} \mid a \otimes f^{\otimes n} = 0 \text{ for some } n \geq 1\}$.*

Proof. Standard techniques verify that the right-hand side is a thick \otimes -ideal and the inclusion then follows from Lemma 3.3; cf. [Bal10a, Theorem 2.15]. \square

Proposition 5.3. *An isomorphism $\alpha : X \xrightarrow{\sim} Y$ in \mathcal{K} induces an isomorphism of rings $\alpha_* : R_X \xrightarrow{\sim} R_Y$. Under this identification, ρ_X coincides with ρ_Y .*

Proof. This is routine from the definitions; cf. Lemma 4.5. \square

Proposition 5.4. *Tensoring on the right by an object Y induces a ring homomorphism $R_X \rightarrow R_{X \otimes Y}$ and a commutative diagram*

$$\begin{array}{ccc} \mathrm{supp}(X) & \xrightarrow{\rho_X} & \mathrm{Spec}(R_X) \\ \uparrow & & \uparrow \\ \mathrm{supp}(X) \cap \mathrm{supp}(Y) & \xrightarrow{\rho_{X \otimes Y}} & \mathrm{Spec}(R_{X \otimes Y}). \end{array} \quad (5.5)$$

If $\mathrm{supp}(X) \subset \mathrm{supp}(Y)$ then the kernel of the map $R_X \rightarrow R_{X \otimes Y}$ consists entirely of nilpotents. There is a similar result for tensoring on the left.

Proof. Note that there is a canonical isomorphism $X^{\otimes n} \otimes Y^{\otimes n} \xrightarrow{\sim} (X \otimes Y)^{\otimes n}$ obtained from the symmetry that preserves the order of the X 's and the order of the Y 's. One can then define a ring homomorphism $E_{X^{\otimes n}} \rightarrow E_{(X \otimes Y)^{\otimes n}}$ as the composition of $- \otimes Y^{\otimes n} : E_{X^{\otimes n}} \rightarrow E_{X^{\otimes n} \otimes Y^{\otimes n}}$ and the isomorphism $E_{X^{\otimes n} \otimes Y^{\otimes n}} \xrightarrow{\sim} E_{(X \otimes Y)^{\otimes n}}$ and one readily verifies that these maps induce a homomorphism $R_X \rightarrow R_{X \otimes Y}$. That (5.5) commutes follows from the definitions, observing that if $\mathcal{P} \in$

$\text{supp}(Y)$ then $a \in \mathcal{P}$ iff $a \otimes Y \in \mathcal{P}$. Finally, if $[f] \in R_X$ is mapped to zero in $R_{X \otimes Y}$ then $X^{\otimes i} \otimes Y^{\otimes j} \otimes f = 0$ for some $i, j \geq 1$. It follows using the condition $\text{supp}(X) \subset \text{supp}(Y)$ that $\text{supp}(X) = \text{supp}(\text{cone}(f))$ and hence that $X^{\otimes k} \in \langle \text{cone}(f) \rangle$ for some $k \geq 1$. Lemma 5.2 then implies that $X^{\otimes k} \otimes f^{\otimes n} = 0$ for some $n \geq 1$ and we conclude that $[f]$ is a nilpotent element of R_X . \square

Proposition 5.6. *For every pair of objects X and Y and every integer $k \geq 1$, tensoring on the left by $X^{\otimes(k-1)}$ induces an isomorphism $R_{X \otimes Y} \xrightarrow{\sim} R_{X^{\otimes k} \otimes Y}$. Under this identification the maps $\rho_{X \otimes Y}$ and $\rho_{X^{\otimes k} \otimes Y}$ coincide.*

Proof. Since the homomorphisms $R_{X \otimes Y} \hookrightarrow R_{X^{\otimes 2} \otimes Y} \hookrightarrow R_{X^{\otimes 3} \otimes Y} \hookrightarrow \dots$ induced by $X \otimes -$ are evidently injective, the problem reduces to showing that $R_{X \otimes Y} \rightarrow R_{X^{\otimes 2} \otimes Y}$ is surjective. In other words, we need to show that every $f \in E_{(X^{\otimes 2} \otimes Y)^{\otimes n}}$ is equivalent in $R_{X^{\otimes 2} \otimes Y}$ to an element coming from $R_{X \otimes Y}$. We'll give the proof under the assumption that $n = 1$. The proof for arbitrary $n \geq 1$ is similar.

Consider the element $g \in E_{(X \otimes Y)^{\otimes 3}}$ corresponding to $X \otimes f \otimes Y^{\otimes 2} \in E_{X^{\otimes 3} \otimes Y^{\otimes 3}}$. Under the map $R_{X \otimes Y} \rightarrow R_{X^{\otimes 2} \otimes Y}$, $[g]$ is sent to the image in $R_{X^{\otimes 2} \otimes Y}$ of the element in $E_{(X^{\otimes 2} \otimes Y)^{\otimes 3}}$ corresponding to $X^{\otimes 4} \otimes f \otimes Y^{\otimes 2} \in E_{X^{\otimes 6} \otimes Y^{\otimes 3}}$. We claim that this element in $E_{(X^{\otimes 3} \otimes Y)^{\otimes 3}}$ equals $(X^{\otimes 2} \otimes Y)^{\otimes 2} \otimes f$ so that the image of $[g]$ in $R_{X^{\otimes 2} \otimes Y}$ equals $[f]$. This is not completely obvious and involves an unilluminating trick. In order to describe this trick, let's write $a := X^{\otimes 2}$ and $b := Y$ for simplicity of notation; so $f \in E_{a \otimes b}$. We will use subscripts to indicate position and we'll drop the tensors from the notation. Consider the diagram

$$\begin{array}{ccc}
a_1 b_1 a_2 b_2 a_3 b_3 & \begin{array}{c} \xrightarrow{ababf} \\ \xrightarrow{abfab} \end{array} & a_1 b_1 a_2 b_2 a_3 b_3 \\
\downarrow \sim & & \downarrow \sim \\
a_1 a_2 b_2 a_3 b_1 b_3 & \begin{array}{c} \xrightarrow{afab^2} \\ \xrightarrow{a^2 bfb} \end{array} & a_1 a_2 b_2 a_3 b_1 b_3 \\
\downarrow \sim & & \downarrow \sim \\
a_1 a_2 a_3 b_1 b_2 b_3 & \xrightarrow{a^2 f b^2} & a_1 a_2 a_3 b_1 b_2 b_3
\end{array}$$

where the vertical maps are the indicated permutations of the factors induced by the symmetry. Note that the composition of the two vertical permutations is the unique permutation from the source to the target that preserves the order of the a 's and the order of the b 's. The top arrow is $(X^2 \otimes Y)^{\otimes 2} \otimes f$ and the bottom arrow is the morphism that it corresponds to in $E_{X^{\otimes 6} \otimes Y^{\otimes 3}}$. The commutativity of the diagram verifies that this equals $X^{\otimes 4} \otimes f \otimes Y^{\otimes 2}$ as claimed. \square

Proposition 5.7. *If \mathcal{K} is a rigid tensor triangulated category then for every object X in \mathcal{K} there is a canonical isomorphism $R_X \simeq R_{DX}$ under which the map ρ_X coincides with ρ_{DX} .*

Proof. The duality functor $D : \mathcal{K}^{\text{op}} \rightarrow \mathcal{K}$ gives a ring isomorphism $[X, X] \xrightarrow{\sim} [DX, DX]^{\text{op}}$ and an easy application of the fact that D is a strong \otimes -functor shows that the isomorphism restricts to an isomorphism $E_X \xrightarrow{\sim} E_{DX}^{\text{op}}$. It is straightforward but tedious to verify that these isomorphisms induce an isomorphism $R_X \xrightarrow{\sim} R_{DX}^{\text{op}} = R_{DX}$. Showing that ρ_X and ρ_{DX} correspond amounts to showing that $\text{supp}(\text{cone}(Df)) = \text{supp}(D(\text{cone}(f)))$. Here we use the fact that D is an exact functor of triangulated categories and the fact that $\text{supp}(DX) = \text{supp}(X)$ in any rigid category. \square

Our next goal is to establish that $\rho_X = \rho_{X \oplus X}$. Note that under the usual identification of $(X \oplus X)^{\otimes n}$ with a \oplus -sum of 2^n copies of $X^{\otimes n}$ an endomorphism $(X \oplus X)^{\otimes n} \rightarrow (X \oplus X)^{\otimes n}$ can be regarded as a $2^n \times 2^n$ matrix (f_{ij}) with entries $f_{ij} : X^{\otimes n} \rightarrow X^{\otimes n}$. We will make such identifications without further comment.

Lemma 5.8. *An endomorphism $f = (f_{ij}) : a_1 \oplus \cdots \oplus a_n \rightarrow a_1 \oplus \cdots \oplus a_n$ is contained in $E_{a_1 \oplus \cdots \oplus a_n}$ if and only if*

- (1) $a_i \otimes f_{jj} = f_{ii} \otimes a_j$ for all i, j , and
- (2) $f_{ij} \otimes (a_1 \oplus \cdots \oplus a_n) = 0$ for $i \neq j$.

Proof. Observe that $f \otimes (a_1 \oplus \cdots \oplus a_n)$, viewed as an $n^2 \times n^2$ matrix, consists of $n \times n$ blocks, each of which is diagonal, while $(a_1 \oplus \cdots \oplus a_n) \otimes f$ consists of $n \times n$ blocks, arranged along the diagonal. Equating the off-diagonal blocks gives the condition that $f_{ij} \otimes a_k = 0$ for all k if $i \neq j$, which is equivalent to condition (2). Similarly, equating the off-diagonals of the diagonal blocks gives the equivalent condition that $a_k \otimes f_{ij} = 0$ for all k if $i \neq j$. On the other hand, the diagonal of the i th diagonal block gives the condition that $f_{ii} \otimes a_j = a_i \otimes f_{jj}$ for all j . \square

Corollary 5.9. *If $f \in E_{(X \oplus X)^{\otimes n}}$ then there exists some $\alpha \in E_{X^{\otimes n}}$ such that $(X \oplus X)^{\otimes n} \otimes f$ (regarded as a matrix of endomorphisms of $X^{\otimes 2n}$) is diagonal with copies of $X^{\otimes n} \otimes \alpha$ along the diagonal.*

Proof. This follows from Lemma 5.8. We can take $\alpha := f_{11}$ for example. \square

Proposition 5.10. *Let \mathcal{K} be a tensor triangulated category and let X be an object in \mathcal{K} . There is a canonical isomorphism $R_X \simeq R_{X \oplus X}$ under which ρ_X coincides with $\rho_{X \oplus X}$.*

Proof. Invoking Lemma 5.8, we see that there is a ring homomorphism $\Delta : E_{X^{\otimes n}} \hookrightarrow E_{(X \oplus X)^{\otimes n}}$ for each $n \geq 1$ which sends $f \in E_{X^{\otimes n}}$ to the diagonal matrix consisting of copies of f on the diagonal. One checks that these maps commute with $X \otimes -$ and $(X \oplus X) \otimes -$ and therefore induce an injection $R_X \hookrightarrow R_{X \oplus X}$. Surjectivity of this map follows from Corollary 5.9. That ρ_X and $\rho_{X \oplus X}$ correspond boils down to the definitions and the fact that $\text{cone}(f \oplus \cdots \oplus f) \simeq \text{cone}(f) \oplus \cdots \oplus \text{cone}(f)$. \square

A similar argument shows that $\rho_X = \rho_{X \oplus \Sigma X}$ after a canonical identification $R_X \simeq R_{X \oplus \Sigma X}$. More generally:

Proposition 5.11. *Let \mathcal{K} be a tensor triangulated category and let X be an object of \mathcal{K} . If Y is a \oplus -sum of suspensions of X then $\rho_X = \rho_Y$ after a canonical identification $R_X \simeq R_Y$.*

Proof. The proof is a more advanced version of the proof of Proposition 5.10. Observe that $(\Sigma^{i_1} X \oplus \cdots \oplus \Sigma^{i_k} X)^{\otimes n}$ may be identified with a \oplus -sum of suspensions of $X^{\otimes n}$. One may define a ‘‘diagonal’’ map $E_{X^{\otimes n}} \hookrightarrow E_{(\Sigma^{i_1} X \oplus \cdots \oplus \Sigma^{i_k} X)^{\otimes n}}$ which sends f to a diagonal matrix whose diagonal entries are copies of f suspended the appropriate numbers of times. One checks that these maps induce a map $R_X \hookrightarrow R_{(\Sigma^{i_1} X \oplus \cdots \oplus \Sigma^{i_k} X)}$ and a similar argument shows that this map is in fact surjective. \square

Remark 5.12. There are graded versions of all of the above results, establishing that ρ_X^\bullet is invariant under suspension, tensor powers, and so on. The only result for which we should be careful is taking duals. The duality induces an isomorphism

$R_X^\bullet \simeq R_{D_X}^{\bullet, \text{op}}$ but we can't remove the op because $R_{D_X}^\bullet$ is only graded-commutative. In any case, there is a canonical homeomorphism $\text{Spec}^h(R_{D_X}^{\bullet, \text{op}}) \simeq \text{Spec}^h(R_{D_X}^\bullet)$ and under these identifications ρ_X^\bullet coincides with $\rho_{D_X}^\bullet$.

Example 5.13. For any object $X \in \mathcal{K}$ there is a homomorphism $[\mathbb{1}, \mathbb{1}]_\bullet \rightarrow E_X^\bullet$ which sends α to $\alpha \otimes X = X \otimes \alpha$. These induce a homomorphism $R_\mathbb{1}^\bullet \rightarrow R_\Phi^\bullet$ for any \otimes -multiplicative set $\Phi \subset \mathcal{K}$ whose kernel consists of nilpotents (cf. Proposition 5.4). If Φ is taken to be the collection of objects that are isomorphic to direct sums of suspensions of $\mathbb{1}$ then this map is an isomorphism (cf. Proposition 5.11). For example, if $\mathcal{K} = D^{\text{perf}}(k)$ for a field k , then every object is a direct sum of suspensions of $\mathbb{1}$ and the ‘‘only’’ comparison map is the original unit comparison map $\rho_\mathbb{1}^\bullet : \text{Spc}(\mathcal{K}) \rightarrow \text{Spec}^h([\mathbb{1}, \mathbb{1}]_\bullet)$. More generally, it would be interesting to know whether $R_\mathbb{1}^\bullet \rightarrow R_\Phi^\bullet$ is an isomorphism (under suitable generation hypotheses) when Φ is the collection of solid objects; in other words, whether the closed set comparison map $\rho_{\text{Spc}(\mathcal{K})}^\bullet$ associated with the whole spectrum reduces to $\rho_\mathbb{1}^\bullet$.

Remark 5.14. Recall from the proof of Theorem 3.8 that under the unnatural comparison map $\rho_{X,A} : \text{supp}(X) \rightarrow \text{Spec}(A)$ the preimage of a Thomason closed subset $V(a_1, \dots, a_n) \subset \text{Spec}(A)$ is exactly the support of the ‘‘Koszul’’ object $\text{cone}(\alpha(a_1)) \otimes \dots \otimes \text{cone}(\alpha(a_n))$. On the other hand, for our natural comparison map $\rho_X : \text{supp}(X) \rightarrow \text{Spec}(R_X)$ the elements of R_X are equivalence classes of endomorphisms, but one still finds that

$$\rho_X^{-1}(V([f_1], \dots, [f_n])) = \text{supp}(\text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n))$$

independent of the choice of representatives f_i . Nevertheless, a different set of representatives $[f'_1], \dots, [f'_n]$ gives a different Koszul object $\text{cone}(f'_1) \otimes \dots \otimes \text{cone}(f'_n)$ and there is no reason *a priori* for the comparison maps of these two Koszul objects to coincide. However, $X^{\otimes i} \otimes \text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n) \simeq X^{\otimes j} \otimes \text{cone}(f'_1) \otimes \dots \otimes \text{cone}(f'_n)$ for some $i, j \geq 1$ and it follows from Proposition 5.6 that the comparison map for $X \otimes \text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n)$ does not depend on the choice of representatives f_i . Thus when one decides to examine a closed set $\text{supp}(X_0)$ more closely by choosing generators of a Thomason closed subset $V([f_1], \dots, [f_n]) \subset \text{Spec}(R_{X_0})$, it is advisable to take the generator of the preimage $\rho_{X_0}^{-1}(V([f_1], \dots, [f_n]))$ to be $X_1 := X_0 \otimes \text{cone}(f_1) \otimes \dots \otimes \text{cone}(f_n)$. A serendipitous consequence of including X_0 as a \otimes -factor is that we then have a ring homomorphism $R_{X_0} \rightarrow R_{X_1}$ and a commutative diagram

$$\begin{array}{ccc} \text{supp}(X_0) & \xrightarrow{\rho_{X_0}} & \text{Spec}(R_{X_0}) \\ \uparrow & & \uparrow \\ & & V([f_1], \dots, [f_n]) \\ \downarrow & & \downarrow \\ \text{supp}(X_1) & \xrightarrow{\rho_{X_1}} & \text{Spec}(R_{X_1}). \end{array}$$

On the other hand, this procedure still apparently depends on the choice of generators for the Thomason closed subset $V([f_1], \dots, [f_n])$.

Idempotent completion. Recall that every tensor triangulated category \mathcal{K} may be embedded into an idempotent-complete tensor triangulated category \mathcal{K}^\natural and that the embedding $i : \mathcal{K} \hookrightarrow \mathcal{K}^\natural$ induces a homeomorphism of spectra (see [Bal05, Remark 3.12]). There is a precise sense in which \mathcal{K} and \mathcal{K}^\natural admit ‘‘the same’’ theory of higher comparison maps. We begin with the following unsurprising result.

Proposition 5.15. *For any non-empty \otimes -multiplicative subset $\Phi \subset \mathcal{K}$, there is a canonical identification $R_{\mathcal{K},\Phi} \simeq R_{\mathcal{K}^\natural,i(\Phi)}$ while $\mathcal{Z}_{\mathcal{K},\Phi} \simeq \mathcal{Z}_{\mathcal{K}^\natural,i(\Phi)}$ under the homeomorphism $i^* : \mathrm{Spc}(\mathcal{K}^\natural) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K})$. Under these identifications, $\rho_{\mathcal{K},\Phi} = \rho_{\mathcal{K}^\natural,i(\Phi)}$.*

Proof. This is a routine verification once one recalls all the definitions. \square

In particular, this tells us that the object comparison maps for objects in \mathcal{K} are unaffected when we pass to \mathcal{K}^\natural . But it still could be possible that in passing to \mathcal{K}^\natural we get new object comparison maps coming from the new objects in \mathcal{K}^\natural . However, this is not the case:

Proposition 5.16. *For any object $X \in \mathcal{K}^\natural$, the object $X \oplus \Sigma X$ is contained in $\mathcal{K} \subset \mathcal{K}^\natural$. There is a canonical isomorphism $R_{\mathcal{K}^\natural,X} \simeq R_{\mathcal{K},X \oplus \Sigma X}$ and after this identification $\rho_{\mathcal{K}^\natural,X} = \rho_{\mathcal{K},X \oplus \Sigma X}$.*

Proof. That $X \oplus \Sigma X$ is contained in \mathcal{K} is a well-known fact; see the proof of [Bal05, Proposition 3.13] for example. Our result then follows from Proposition 5.15 together with the result of Proposition 5.11 which told us that $\rho_X = \rho_{X \oplus \Sigma X}$. \square

Next we can ask about the closed set comparison maps.

Proposition 5.17. *Let \mathcal{Z} be a closed subset of $\mathrm{Spc}(\mathcal{K})$ and let $\mathcal{Z}' = (i^*)^{-1}(\mathcal{Z})$ be the corresponding closed subset of $\mathrm{Spc}(\mathcal{K}^\natural)$. There is a canonical isomorphism $R_{\mathcal{K},\mathcal{Z}} \simeq R_{\mathcal{K}^\natural,\mathcal{Z}'}$ such that with the identification $\mathcal{Z} \simeq \mathcal{Z}'$ the comparison map $\rho_{\mathcal{K},\mathcal{Z}}$ coincides with $\rho_{\mathcal{K}^\natural,\mathcal{Z}'}$.*

Proof. Let $\Phi = \{x \in \mathcal{K} \mid \mathrm{supp}_{\mathcal{K}}(x) \supset \mathcal{Z}\}$ and let $\Phi' = \{a \in \mathcal{K}^\natural \mid \mathrm{supp}_{\mathcal{K}^\natural}(a) \supset \mathcal{Z}'\}$. Then $i(\Phi) \subset \Phi'$ and we have an induced ring homomorphism $R_{i(\Phi)} \rightarrow R_{\Phi'}$. If $a \in \mathcal{K}^\natural$ then $a \oplus \Sigma a \in i(\mathcal{K})$ as before and it follows that if $a \in \Phi'$ then $a \oplus \Sigma a \in i(\Phi)$. We claim that for any $a \in \Phi'$, the diagram

$$\begin{array}{ccc} R_{i(\Phi)} & \longrightarrow & R_{\Phi'} \\ \uparrow & \nearrow & \uparrow \\ R_{a \oplus \Sigma a} & \xleftarrow{\sim} & R_a \end{array}$$

commutes, where the bottom row is the isomorphism obtained in Proposition 5.11; it will follow that the map $R_{i(\Phi)} \rightarrow R_{\Phi'}$ is surjective. The commutativity of the top triangle is immediate. On the other hand, consider some $[f] \in R_a$, say with $f \in E_{a \otimes^n}$. Suppose for starters that $n = 1$. Then $[f]$ maps to $[f \oplus \Sigma f]$ in $R_{a \oplus \Sigma a}$ and so the question (in this case) is whether $[f \oplus \Sigma f] = [f]$ in $R_{\Phi'}$. This is readily verified:

$$\begin{aligned} [f \oplus \Sigma f] &= [(f \oplus \Sigma f) \otimes a] = [(f \otimes a) \oplus (\Sigma f \otimes a)] = \\ &= [(a \otimes f) \oplus (\Sigma a \otimes f)] = [(a \oplus \Sigma a) \otimes f] = [f]. \end{aligned}$$

For general $n \geq 1$, regarding $(a \oplus \Sigma a)^{\otimes n}$ as a \oplus -sum of suspensions of $a \otimes a$, an element $f \in E_{a \otimes^n}$ is sent to a \oplus -sum of suspensions of f and it comes down to showing that $[f] = [\Sigma^{i_1} f \oplus \Sigma^{i_2} f \oplus \cdots \oplus \Sigma^{i_k} f]$ in $R_{\Phi'}$ (which can be verified in a similar manner).

On the other hand, if $[f] \in R_{i(\Phi)}$ is an element that is sent to zero in $R_{\Phi'}$, then $a \otimes f = 0$ for some $a \in \Phi'$ and $(a \oplus \Sigma a) \otimes f \simeq (a \otimes f) \oplus \Sigma(a \otimes f) = 0$ shows that $[f] = 0$ in $R_{i(\Phi)}$. Therefore the map $R_{i(\Phi)} \rightarrow R_{\Phi'}$ is also injective. It is clear that

under this isomorphism $\rho_{i(\Phi)} = \rho_{\Phi'}$ while Proposition 5.15 implies that $\rho_{\Phi} = \rho_{i(\Phi)}$ after identifying $\mathcal{Z} \simeq \mathcal{Z}'$. \square

In other words, we have established that \mathcal{K} and \mathcal{K}^{\natural} give precisely the same object comparison maps and precisely the same closed set comparison maps.

6. TOPOLOGICAL RESULTS

Throughout this section let \mathcal{K} be a tensor triangulated category and let $\Phi \subset \mathcal{K}$ be a non-empty set of objects closed under the \otimes -product.

Proposition 6.1. *If \mathcal{Z}_{Φ} is connected then $\text{Spec}(R_{\Phi})$ is connected.*

Proof. By the results of the last section it suffices to prove the result under the additional hypothesis that \mathcal{K} is idempotent-complete. If $\text{Spec}(R_{\Phi})$ is disconnected then there is a non-trivial idempotent $[e_X]$ in the ring R_{Φ} . By Lemma 4.8, $[e_X] = [e_X]^2 = [e_X^2]$ implies that there is an $a \in \Phi$ such that $e_X \otimes a \otimes X = X \otimes a \otimes e_X^2 = (X \otimes a \otimes e_X)^2$ while $e_X \otimes a \otimes X = X \otimes a \otimes e_X$ since e_X is \otimes -balanced. Moreover, $[e_X] \neq 0$ implies $X \otimes a \otimes e_X \neq 0$ and $[e_X] \neq 1$ implies $X \otimes a \otimes e_X \neq \text{id}_{X \otimes a \otimes X}$, so $f := X \otimes a \otimes e_X$ is a non-trivial idempotent endomorphism of $X \otimes a \otimes X$. Moreover, it is contained in $E_{X \otimes a \otimes X}$ and hence gives an element $[f]$ of R_{Φ} .

Since \mathcal{K} is idempotent-complete, there is an associated decomposition $X \otimes a \otimes X \simeq a_1 \oplus a_2$ for two non-zero objects a_1 and a_2 such that f becomes the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In particular, $\text{cone}(f) \simeq \Sigma a_2 \oplus a_2$ and $\text{cone}(\text{id}_{X \otimes a \otimes X} - f) \simeq \Sigma a_1 \oplus a_1$. It follows that $\mathcal{Z}_{\Phi} \cap \text{supp}(a_1) \cap \text{supp}(a_2) = \emptyset$ because otherwise there would be a $\mathcal{P} \in \mathcal{Z}_{\Phi}$ with $\text{cone}(f) \notin \mathcal{P}$ and $\text{cone}(\text{id}_{X \otimes a \otimes X} - f) \notin \mathcal{P}$, which would imply that both $[f]$ and $1 - [f]$ are contained in the prime $\rho_{\Phi}(\mathcal{P})$. We conclude that $\mathcal{Z}_{\Phi} = \mathcal{Z}_{\Phi} \cap \text{supp}(X \otimes a \otimes X) = (\mathcal{Z}_{\Phi} \cap \text{supp}(a_1)) \sqcup (\mathcal{Z}_{\Phi} \cap \text{supp}(a_2))$ is a disjoint union of closed sets, and it remains to show that $\mathcal{Z}_{\Phi} \cap \text{supp}(a_i) \neq \emptyset$ for $i = 1, 2$.

If $\mathcal{Z}_{\Phi} \cap \text{supp}(a_1)$ were empty then the quasi-compactness of $\text{Spc}(\mathcal{K})$ would imply that there is a $c \in \Phi$ such that $\text{supp}(c) \cap \text{supp}(a_1) = \emptyset$ and hence that $c \otimes a_1$ is \otimes -nilpotent. But observe that under the identification $X \otimes a \otimes X \simeq a_1 \oplus a_2$, the endomorphism $f^{\otimes n}$ becomes a matrix with all zero entries except for $\text{id}_{a_1^{\otimes n}}$ at one position along the diagonal. From this it is clear that $c^{\otimes n} \otimes f^{\otimes n} = 0$ for some $n \geq 1$ since $c \otimes a_1$ is \otimes -nilpotent. But then $[e_X] = [f] = [f^n] = [f^{\otimes n}] = [c^{\otimes n} \otimes f^{\otimes n}] = 0$ in the ring R_{Φ} , which contradicts the fact that $[e_X]$ is nontrivial. A similar argument shows that if $\mathcal{Z}_{\Phi} \cap \text{supp}(a_2)$ were empty then $[e_X] = 1$. \square

For the converse of the above result, we need to add additional assumptions.

Proposition 6.2. *Suppose \mathcal{K} is rigid and \mathcal{Z}_{Φ} is Thomason. If $\text{Spec}(R_{\Phi})$ is connected then \mathcal{Z}_{Φ} is connected.*

Proof. By passing to the idempotent completion it suffices to prove the result under the additional hypothesis that \mathcal{K} is idempotent-complete (cf. the results in the last section and note that \mathcal{K} rigid implies that \mathcal{K}^{\natural} is also rigid [Bal07, Proposition 2.15(i)]). Suppose $\mathcal{Z}_{\Phi} = Y_1 \sqcup Y_2$ is a disjoint union of non-empty closed sets. Each Y_i is quasi-compact (being closed) and it follows from the fact that they are disjoint and that \mathcal{Z}_{Φ} is Thomason that each Y_i is Thomason. It also follows from the definition $\mathcal{Z}_{\Phi} := \bigcap_{X \in \Phi} \text{supp}(X)$ and the fact that $\text{Spc}(\mathcal{K}) \setminus \mathcal{Z}_{\Phi}$ is quasi-compact that $\mathcal{Z}_{\Phi} = \text{supp}(a)$ for some $a \in \Phi$. Since \mathcal{K} is rigid and idempotent-complete, the generalized Carlson theorem [Bal07, Remark 2.12] implies that there

exist $a_1, a_2 \in \mathcal{K}$ such that $a \simeq a_1 \oplus a_2$ and $\text{supp}(a_i) = Y_i$ for $i = 1, 2$. Since \mathcal{K} is rigid and $\text{supp}(a_1) \cap \text{supp}(a_2) = \emptyset$ it follows [Bal07, Corollary 2.8] that $[a_i, a_j] = 0$ for $i \neq j$. This implies that the idempotent $f := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is \otimes -balanced ($f \otimes a = a \otimes f$) and hence provides an idempotent element $[f]$ of the ring R_Φ . If $[f]$ was a trivial idempotent it would follow that $b \otimes a_1 = 0$ or $b \otimes a_2 = 0$ for some $b \in \Phi$. But $Y_i = \text{supp}(a_i) = \text{supp}(b \otimes a_i)$ since $\text{supp}(a_i) \subset \mathcal{Z}_\Phi \subset \text{supp}(b)$ and so $b \otimes a_i = 0$ contradicts the fact that Y_i is non-empty. We conclude that $[f]$ is a nontrivial idempotent of the commutative ring R_Φ and hence that $\text{Spec}(R_\Phi)$ is disconnected. \square

Remark 6.3. Example 6.13 below will provide some (non-rigid) examples of tensor triangulated categories for which the conclusion of Proposition 6.2 does not hold.

Remark 6.4. For any (graded-)commutative graded ring R^\bullet , the space $\text{Spec}^h(R^\bullet)$ is connected if and only if $\text{Spec}(R^0)$ is connected, so the graded version of the above statements also holds.

Lemma 6.5. *If c is an object in \mathcal{K} with $\mathcal{Z}_\Phi \subset \text{supp}(c)$ then there is some $a \in \Phi$ such that $\text{supp}(a) \subset \text{supp}(c)$.*

Proof. Since $\text{Spc}(\mathcal{K}) \setminus \text{supp}(c)$ is quasi-compact (recall Lemma 2.3), it follows from the definition $\mathcal{Z}_\Phi := \bigcap_{X \in \Phi} \text{supp}(X)$ that there is a finite collection of objects $X_1, \dots, X_n \in \Phi$ such that $\text{supp}(X_1) \cap \dots \cap \text{supp}(X_n) \subset \text{supp}(c)$ and we can take $a := X_1 \otimes \dots \otimes X_n \in \Phi$. \square

Proposition 6.6. *Consider the map $\rho_\Phi^\bullet : \mathcal{Z}_\Phi \rightarrow \text{Spec}^h(R_\Phi^\bullet)$ and any closed set $\mathcal{W} \subset \text{Spec}^h(R_\Phi^\bullet)$. If $(\rho_\Phi^\bullet)^{-1}(\mathcal{W}) = \mathcal{Z}_\Phi$ then $\mathcal{W} = \text{Spec}^h(R_\Phi^\bullet)$. In other words, the preimage of a proper closed set remains proper.*

Proof. Let $\mathcal{W} = V(I)$ for some homogeneous ideal $I \subset R_\Phi^\bullet$ and let $[f]$ be an arbitrary homogeneous element of I . It follows from our hypothesis that $[f] \in \rho_\Phi^\bullet(\mathcal{P})$ for every $\mathcal{P} \in \mathcal{Z}_\Phi$ and hence that $\mathcal{Z}_\Phi \subset \text{supp}(\text{cone}(f))$. Lemma 6.5 then implies that there is some $a \in \Phi$ such that $\text{supp}(a) \subset \text{supp}(\text{cone}(f))$. It follows that $a^{\otimes i} \in \langle \text{cone}(f) \rangle$ for some $i \geq 1$ and hence that $a^{\otimes i} \otimes f^{\otimes j} = 0$ for some $i, j \geq 1$ by Lemma 5.2. But then $[f]^j = [f^{\otimes j}] = [a^{\otimes i} \otimes f^{\otimes j}] = 0$ so that $[f]$ is a nilpotent element of R_Φ . Since every homogeneous element of I is nilpotent, $\mathcal{W} = V(I) = \text{Spec}^h(R_\Phi^\bullet)$. \square

Remark 6.7. One easily verifies that the ungraded maps ρ_Φ also have the property that preimages of proper closed subsets remain proper, either by carrying out an ungraded version of the above proof, or as a corollary of the above proposition by observing that the map $(-)^0 : \text{Spec}^h(R_\Phi^\bullet) \rightarrow \text{Spec}(R_\Phi)$ has this property.

Corollary 6.8. *The maps $\rho_\Phi^\bullet : \mathcal{Z}_\Phi \rightarrow \text{Spec}(R_\Phi^\bullet)$ and $\rho_\Phi : \mathcal{Z} \rightarrow \text{Spec}(R_\Phi)$ have dense images. Consequently, if \mathcal{Z}_Φ is irreducible then $\text{Spec}(R_\Phi^\bullet)$ and $\text{Spec}(R_\Phi)$ are also irreducible.*

Lemma 6.9. *If \mathcal{K} is rigid and $[f]$ is a homogeneous non-unit in the ring R_Φ^\bullet then there is some $\mathcal{P} \in \mathcal{Z}_\Phi$ such that $[f] \in \rho_\Phi^\bullet(\mathcal{P})$.*

Proof. If there were no such \mathcal{P} then $\mathcal{Z}_\Phi \cap \text{supp}(\text{cone}(f)) = \emptyset$ which implies by the quasi-compactness of $\text{Spc}(\mathcal{K})$ that $\text{supp}(a) \cap \text{supp}(\text{cone}(f)) = \emptyset$ for some $a \in \Phi$. But $\text{supp}(a \otimes \text{cone}(f)) = \emptyset$ implies by the rigidity of \mathcal{K} that $a \otimes \text{cone}(f) = 0$. Thus $a \otimes f$ is invertible and hence $[f]$ is a unit in R_Φ^\bullet . \square

Proposition 6.10. *Suppose \mathcal{K} is rigid. If the map $\rho_\Phi : \mathcal{Z}_\Phi \rightarrow \mathrm{Spec}(R_\Phi)$ is a constant map then $\mathrm{Spec}(R_\Phi)$ is a point. Similarly, if the map $\rho_\Phi^\bullet : \mathcal{Z}_\Phi \rightarrow \mathrm{Spec}^h(R_\Phi^\bullet)$ is a constant map then $\mathrm{Spec}^h(R_\Phi^\bullet)$ is a point.*

Proof. Suppose \mathfrak{p} is a prime in R_Φ such that $\rho_\Phi(\mathcal{P}) = \mathfrak{p}$ for all $\mathcal{P} \in \mathcal{Z}_\Phi$. Lemma 6.9 then implies that \mathfrak{p} contains every non-unit of R_Φ . It follows that R_Φ is local with \mathfrak{p} its unique maximal ideal. On the other hand, Corollary 6.8 implies that $\{\mathfrak{p}\}$ is dense, and so $\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}} = \mathrm{Spec}(R_\Phi)$. An identical argument works in the graded case. \square

Remark 6.11. The theorems proved in this section are new even in the case of Balmer’s original map $\rho_\mathbb{1}$. For example, Corollary 6.8 implies that $\rho_\mathbb{1}$ always has dense image without any assumptions on the category \mathcal{K} . This result is particularly interesting in light of the surjectivity criteria for $\rho_\mathbb{1}$ established in [Bal10a].

Remark 6.12. One strategy for studying the spectrum is to iteratively build a filtration of closed subsets by pulling back filtrations via our closed set comparison maps. More precisely, we begin with the trivial filtration $\{\mathrm{Spc}(\mathcal{K})\}$ and at each iterative step we consider every closed set \mathcal{Z} in the filtration (or only those that were newly added at the last step) and refine the filtration below \mathcal{Z} by pulling back the filtration of closed subsets of $\mathrm{Spec}^h(R_\mathcal{Z}^\bullet)$ via the map $\rho_\mathcal{Z}^\bullet : \mathcal{Z} \rightarrow \mathrm{Spec}^h(R_\mathcal{Z}^\bullet)$. A key result for this idea is Proposition 6.6 which asserts that proper closed subsets of $\mathrm{Spec}^h(R_\mathcal{Z}^\bullet)$ pull back to proper subsets of \mathcal{Z} . This implies that the process continues to refine the spectrum for as long as the spaces $\mathrm{Spec}^h(R_\mathcal{Z}^\bullet)$ are non-trivial. However, an obstacle is the possibility that $\mathrm{Spec}^h(R_\mathcal{Z}^\bullet)$ might be trivial for some non-trivial closed set \mathcal{Z} , in which case the internal structure of \mathcal{Z} would remain hidden. In fact, there are examples of tensor triangulated categories for which it seems the process may hit the wall at the very first step (cf. Example 6.13 below). Nevertheless, these examples are non-rigid and there is some hope that under suitable hypotheses this obstacle might disappear.

Example 6.13. Let P be a finite poset, k a field, and let $\mathcal{K} := D^b(\mathrm{rep}_k(P))$ be the derived category of finite-dimensional k -linear representations of P . The abelian category $\mathrm{rep}_k(P)$ has an exact vertex-wise \otimes -structure and \mathcal{K} inherits the structure of a tensor triangulated category. Recognizing that representations of P are the same thing as quiver representations of the associated Hasse diagram with full commutativity relations, one sees that the work of [LS] completely describes the spectrum $\mathrm{Spc}(\mathcal{K})$. It turns out to be rather trivial: a discrete space with points corresponding to the elements of P . For a representation V regarded as a complex concentrated in degree 0, $\mathrm{supp}(V) = \{x \in P \mid V_x \neq 0\}$. If P has a least element then $\mathbb{1}$ is projective (so $[\mathbb{1}, \mathbb{1}]_i = 0$ for $i \neq 0$) and $[\mathbb{1}, \mathbb{1}] \simeq k$ by inspection. There are thus many examples where $\mathrm{Spec}^h(R_\mathbb{1}^\bullet)$ is trivial (in particular, connected) but $\mathrm{Spc}(\mathcal{K})$ is disconnected. This doesn’t contradict Proposition 6.2 because these examples of derived quiver representations are not rigid. For example, consider the simplest non-trivial example: $P = (1 \rightarrow 2)$. Let $S_1 = (k \rightarrow 0)$ and $S_2 = (0 \rightarrow k)$ be the two simple representations. There is an obvious exact triangle $S_2 \rightarrow \mathbb{1} \rightarrow S_1 \rightarrow \Sigma S_2$. If \mathcal{K} were rigid then the fact that $\mathrm{supp}(S_1) \cap \mathrm{supp}(\Sigma S_2) = \mathrm{supp}(S_1) \cap \mathrm{supp}(S_2) = \emptyset$ implies that $[S_1, \Sigma S_2] = 0$, and hence that the exact triangle splits: $\mathbb{1} \simeq S_1 \oplus S_2$ which is evidently not the case.

7. TRIANGULAR LOCALIZATION

An important feature of the original unit comparison maps developed in [Bal10a] is that it is possible to localize the category \mathcal{K} with respect to primes in the ring $[\mathbb{1}, \mathbb{1}]$, which enables one to reduce questions about the unit comparison maps to the case where the target ring is local. Fortunately, one may establish such a localization technique for our more general comparison maps.

Theorem 7.1. *Let \mathcal{K} be a tensor triangulated category, $\Phi \subset \mathcal{K}$ a non-empty set of objects closed under the \otimes -product, $S \subset R_{\mathcal{K}, \Phi}^\bullet$ a multiplicative set of even (hence central) homogeneous elements, and $q : \mathcal{K} \rightarrow \mathcal{K}/\mathcal{J}$ the Verdier quotient of \mathcal{K} with respect to the thick \otimes -ideal $\mathcal{J} := \langle \text{cone}(s) \mid [s] \in S \rangle$. Then $R_{\mathcal{K}/\mathcal{J}, \Phi}^\bullet$ is isomorphic to the ring-theoretic localization $S^{-1}(R_{\mathcal{K}, \Phi}^\bullet)$ and we have a diagram*

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{K}/\mathcal{J}, \Phi} \hookrightarrow & \longrightarrow & \mathcal{Z}_{\mathcal{K}, \Phi} \\ \downarrow \rho_{\mathcal{K}/\mathcal{J}, \Phi}^\bullet & & \downarrow \rho_{\mathcal{K}, \Phi}^\bullet \\ \text{Spec}^h(R_{\mathcal{K}/\mathcal{J}, \Phi}^\bullet) = \text{Spec}^h(S^{-1}R_{\mathcal{K}, \Phi}^\bullet) \hookrightarrow & \longrightarrow & \text{Spec}^h(R_{\mathcal{K}, \Phi}^\bullet). \end{array} \quad (7.2)$$

that is commutative and cartesian: $\mathcal{Z}_{\mathcal{K}/\mathcal{J}, \Phi} \cong \{\mathcal{P} \in \mathcal{Z}_{\mathcal{K}, \Phi} \mid \rho_{\mathcal{K}, \Phi}^\bullet(\mathcal{P}) \cap S = \emptyset\}$.

Remark 7.3. If $S \subset R^0$ then $(S^{-1}R^\bullet)^0 = S^{-1}R^0$ and one readily verifies that applying $(-)^0$ to the bottom row of (7.2) yields a commutative, cartesian diagram

$$\begin{array}{ccc} \mathcal{Z}_{\mathcal{K}/\mathcal{J}, \Phi} \hookrightarrow & \longrightarrow & \mathcal{Z}_{\mathcal{K}, \Phi} \\ \downarrow \rho_{\mathcal{K}/\mathcal{J}, \Phi} & & \downarrow \rho_{\mathcal{K}, \Phi} \\ \text{Spec}^h(R_{\mathcal{K}/\mathcal{J}, \Phi}) = \text{Spec}^h(S^{-1}R_{\mathcal{K}, \Phi}) \hookrightarrow & \longrightarrow & \text{Spec}^h(R_{\mathcal{K}, \Phi}). \end{array} \quad (7.4)$$

This gives the ungraded version of our localization result.

The remainder of this section is devoted to proving the theorem. For purposes of clarity we will prove the ungraded version—the graded result stated in Theorem 7.1 is proved in the same way but the notation gets more cumbersome. Thus, for the rest of the section we fix a multiplicative set $S \subset R_{\mathcal{K}, \Phi}$. For an element $[s] \in S$, we'll use the notation X_s to indicate that the representative s is an endomorphism of X_s . For morphisms in \mathcal{K}/\mathcal{J} we'll use the left fractions of [Kra10, Section 3]. It is immediate from the definition of \mathcal{J} that the canonical map $R_{\mathcal{K}, \Phi} \rightarrow R_{\mathcal{K}/\mathcal{J}, q(\Phi)}$ factors as

$$R_{\mathcal{K}, \Phi} \xrightarrow{\epsilon} S^{-1}(R_{\mathcal{K}, \Phi}) \xrightarrow{i} R_{\mathcal{K}/\mathcal{J}, q(\Phi)}$$

where ϵ is the canonical localization map. In order to show that i is an isomorphism we will need the following three lemmas:

Lemma 7.5. *If $a \in \mathcal{J}$ then there is a representative s of an element $[s] \in S$ with the property that $a \otimes s = 0$.*

Proof. One verifies using standard techniques that the collection of objects $a \in \mathcal{K}$ for which there is a representative s of an element $[s] \in S$ such that $a \otimes s^{\otimes n} = 0$ for some $n \geq 1$ forms a thick \otimes -ideal of \mathcal{K} . It contains each $\text{cone}(s)$ by Lemma 3.3 and hence it contains \mathcal{J} . Since S is multiplicative, $[s^{\otimes n}] = [s]^n \in S$. \square

Lemma 7.6. *If $f : a \rightarrow b$ and $g : c \rightarrow d$ are morphisms such that $\text{cone}(f) \otimes g = 0$ then there exists a morphism $u : b \otimes c \rightarrow a \otimes d$ such that $a \otimes g = u \circ (f \otimes c)$ and a morphism $v : b \otimes c \rightarrow a \otimes d$ such that $b \otimes g = (f \otimes d) \circ v$.*

Proof. The morphisms u and v are obtained from the morphism of exact triangles

$$\begin{array}{ccccccc}
a \otimes c & \xrightarrow{f \otimes c} & b \otimes c & \longrightarrow & \text{cone}(f) \otimes c & \longrightarrow & \Sigma(a \otimes c) \\
\downarrow a \otimes g & \nearrow u & \downarrow b \otimes g & & \downarrow \text{cone}(f) \otimes g = 0 & & \downarrow \Sigma(a \otimes g) \\
a \otimes d & \xrightarrow{f \otimes d} & b \otimes d & \longrightarrow & \text{cone}(f) \otimes d & \longrightarrow & \Sigma(a \otimes d) \\
& & \nwarrow v & & & &
\end{array}$$

by the weak kernel and cokernel properties of exact triangles. \square

Lemma 7.7. *Let $f : a \rightarrow a$ be an endomorphism in a tensor triangulated category. If there exists a \otimes -balanced automorphism $\sigma : a \rightarrow a$ such that $f \circ \sigma$ is \otimes -balanced then f is \otimes -balanced.*

Proof. This is easily verified from the definitions recalling that E_a is an inverse closed subring of $[a, a]$. \square

Proposition 7.8. *The map $i : S^{-1}(R_{\mathcal{K}, \Phi}) \rightarrow R_{\mathcal{K}/\mathcal{J}, q(\Phi)}$ is injective.*

Proof. Consider an element $[f]/[s] \in S^{-1}(R_{\mathcal{K}, \Phi})$. If $i([f]/[s]) = 0$ then

$$[(X_f \otimes X_s \xrightarrow{f \otimes 1} X_f \otimes X_s \xleftarrow{1 \otimes s} X_f \otimes X_s)] = 0$$

in $R_{\mathcal{K}/\mathcal{J}, q(\Phi)}$ which implies that there is an $a \in \Phi$ such that

$$(a \otimes X_f \otimes X_s \xrightarrow{1 \otimes f \otimes 1} a \otimes X_f \otimes X_s \xleftarrow{1 \otimes s \otimes 1} a \otimes X_f \otimes X_s) = 0$$

as a morphism in \mathcal{K}/\mathcal{J} . It follows that there is a morphism $k : a \otimes X_f \otimes X_s \rightarrow b$ in \mathcal{K} with $\text{cone}(k) \in \mathcal{J}$ such that $k \circ (a \otimes f \otimes X_s) = 0$. By Lemma 7.5 there is some $[t] \in S$ such that $\text{cone}(k) \otimes t = 0$ and hence by Lemma 7.6 there is a morphism $u : b \otimes X_t \rightarrow a \otimes X_f \otimes X_s \otimes X_t$ such that $u \circ (k \otimes X_t) = a \otimes X_f \otimes X_s \otimes t$. In the ring $R_{\mathcal{K}, \Phi}$ we then have

$$[t] \cdot [f] = [(a \otimes X_f \otimes X_s \otimes t) \circ (a \otimes f \otimes X_s \otimes X_t)] = 0$$

and we conclude that $[f]/[s] = ([t] \cdot [f])/([t] \cdot [s]) = 0$ in $S^{-1}(R_{\mathcal{K}, \Phi})$. \square

Proposition 7.9. *The map $i : S^{-1}(R_{\mathcal{K}, \Phi}) \rightarrow R_{\mathcal{K}/\mathcal{J}, \Phi}$ is surjective.*

Proof. Consider an element $[(a \xrightarrow{f} b \xleftarrow{\sigma} a)] \in R_{\mathcal{K}/\mathcal{J}, q(\Phi)}$. By Lemma 7.5 there is some $[s] \in S$ such that $\text{cone}(\sigma) \otimes s = 0$. It then follows from Lemma 7.6 that there exists a morphism $u : b \otimes X_s \rightarrow a \otimes X_s$ such that $u \circ (\sigma \otimes X_s) = a \otimes s$. We thus have an equality of left fractions

$$(a \otimes X_s \xrightarrow{f \otimes 1} b \otimes X_s \xleftarrow{\sigma \otimes 1} a \otimes X_s) = (a \otimes X_s \xrightarrow{u \circ (f \otimes 1)} a \otimes X_s \xleftarrow{1 \otimes s} a \otimes X_s).$$

The left-hand side is an element of $E_{\mathcal{K}/\mathcal{J}, a \otimes X_s}$ and so it follows from Lemma 7.7 that

$$(a \otimes X_s \xrightarrow{u \circ (f \otimes 1)} a \otimes X_s \xleftarrow{\text{id}} a \otimes X_s)$$

is an element of $E_{\mathcal{K}/\mathcal{J}, X \otimes X_s}$. The claim then follows from Lemma 7.10 below. \square

Lemma 7.10. *If $f : a \rightarrow a$ is an endomorphism of an object $a \in \Phi$ such that $q(f)$ is \otimes -balanced as an endomorphism in \mathcal{K}/\mathcal{J} then $[q(f)] \in R_{\mathcal{K},q(\Phi)}$ is contained in the image of $i : S^{-1}(R_{\mathcal{K},\Phi}) \rightarrow R_{\mathcal{K}/\mathcal{J},\Phi}$.*

Proof. It follows from the equality of fractions

$$(a \otimes a \xrightarrow{a \otimes f} a \otimes a \xleftarrow{\text{id}} a \otimes a) = (a \otimes a \xrightarrow{f \otimes a} a \otimes a \xleftarrow{\text{id}} a \otimes a)$$

that there is a morphism $\tau : a \otimes a \rightarrow b$ in \mathcal{K} such that $\text{cone}(\tau) \in \mathcal{J}$ and $\tau \circ (a \otimes f) = \tau \circ (f \otimes a)$. By Lemma 7.5 and Lemma 7.6 there is a $[t] \in S$ such that $\text{cone}(\tau) \otimes t = 0$ and a morphism $u : b \otimes X_t \rightarrow a \otimes a \otimes X_t$ such that $u \circ (\tau \otimes X_t) = a \otimes a \otimes t$. It follows that $a \otimes f \otimes t = f \otimes a \otimes t$ and we conclude using Lemma 4.1 that $f \otimes t$ is an element of $E_{a \otimes X_t}$. Thus

$$[(a \xrightarrow{f} a \xleftarrow{\text{id}} a)] \cdot [(X_t \xrightarrow{t} X_t \xleftarrow{\text{id}} X_t)] = [(a \otimes X_t \xrightarrow{f \otimes t} a \otimes X_t \xleftarrow{\text{id}} a \otimes X_t)]$$

is contained in the image of i . \square

8. STABLE HOMOTOPY THEORY

Let SH denote the stable homotopy category of spectra and let SH^{fin} denote its full subcategory of finite spectra. For a fixed prime p , one usually defines the stable homotopy category of finite p -local spectra $\text{SH}_{(p)}^{\text{fin}}$ as the full subcategory of SH consisting of all spectra isomorphic to the p -localization of a finite spectrum, but for our purposes it is convenient to recognize that this is equivalent to the Verdier quotient of SH^{fin} by the finite p -acyclic spectra.

The purpose of this section is to illustrate the iterative method for examining fibers of comparison maps in the example of the stable homotopy category of finite spectra SH^{fin} . This will depend on a description of the graded centers of endomorphism rings of finite p -local spectra provided by [HS98] which affords a description of the ring

$$A_X^\bullet := \text{Center}([X, X]_\bullet) \cap E_X^\bullet$$

for every finite p -local spectrum X , but not of the non-commutative ring E_X^\bullet , nor the ring $R_X^\bullet = \text{colim}_{n \geq 1} E_X^{\bullet \otimes n}$. For this reason, we'll have to settle for the unnatural comparison maps of Section 3.

First, let us briefly recall the classification of thick subcategories in $\text{SH}_{(p)}^{\text{fin}}$. For each $n \geq 1$, the n th Morava K -theory spectrum $K(n)$ is a p -local ring spectrum whose coefficient ring $K(n)_\bullet := \pi_\bullet(K(n))$ is the graded-field $\mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. The collection of Morava K -theories is completed by $K(0) := H\mathbb{Q}$ and $K(\infty) := H\mathbb{F}_p$ and for each $0 \leq n \leq \infty$ provides a stable homological functor

$$K(n)_\bullet(-) : \text{SH}_{(p)}^{\text{fin}} \rightarrow K(n)_\bullet\text{-grMod}$$

which is in fact a strong \otimes -functor. It follows, using the fact that $K(n)_\bullet$ is a graded-field, that the kernel of this functor is more than a thick subcategory—it is a prime \otimes -ideal; i.e. a point of $\text{Spc}(\text{SH}_{(p)}^{\text{fin}})$. Conforming to the notation of [HS98] we define $\mathcal{C}_0 := \text{SH}_{(p)}^{\text{fin}}$, $\mathcal{C}_n := \ker(K(n-1)_\bullet(-))$ for $n \geq 1$, and $\mathcal{C}_\infty := \ker(K(\infty)_\bullet(-))$. These categories fit into a filtration

$$0 = \mathcal{C}_\infty \subsetneq \cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0 = \text{SH}_{(p)}^{\text{fin}}$$

and the Hopkins-Smith classification theorem asserts that *every* thick \otimes -ideal is of the form \mathcal{C}_n for some $0 \leq n \leq \infty$. From a different point of view, this shows that $\mathrm{Spc}(\mathrm{SH}_{(p)}^{\mathrm{fin}})$ consists of a sequence of points

$$\mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \mathcal{C}_3 \rightarrow \cdots \rightarrow \mathcal{C}_n \rightarrow \mathcal{C}_{n+1} \rightarrow \cdots \rightarrow \mathcal{C}_\infty = (0)$$

where \rightarrow indicates specialization: $\overline{\{\mathcal{C}_n\}} = \{\mathcal{C}_i \mid i \geq n\}$.

The main results from [HS98] which allow for a description of our higher comparison maps arise from their study of non-nilpotent (graded) endomorphisms of finite p -local spectra. Recall that one statement of the Nilpotence Theorem is that an endomorphism $f : \Sigma^d X \rightarrow X$ of a finite p -local spectrum is nilpotent iff $K(n)_\bullet(f)$ is nilpotent for all $0 \leq n < \infty$. This motivates the following definition which aims to pin down those non-nilpotent endomorphisms that are as simple as possible.

Definition 8.1. [HS98, Definition 8] Let $n \geq 1$ and let X be a finite p -local spectrum. An endomorphism $f : \Sigma^d X \rightarrow X$ is a v_n -selfmap if $K(n)_\bullet(f)$ is an isomorphism, and $K(i)_\bullet(f)$ is nilpotent for $i \neq n$.

It follows from the definitions (and the Nilpotence Theorem) that if $X \in \mathcal{C}_{n+1}$ then v_n -selfmaps are the same thing as nilpotent selfmaps but that if $X \notin \mathcal{C}_{n+1}$ then v_n -selfmaps are never nilpotent. It is also easily shown that $X \in \mathcal{C}_n$ is a necessary condition for the existence of a v_n -selfmap. Thus, the notion is mostly of interest for $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$. The first result of substance is that v_n -selfmaps exist:

Theorem 8.2. [HS98, Theorem 9] *A finite p -local spectrum X admits a v_n -selfmap if and only if $X \in \mathcal{C}_n$.*

The most important properties about v_n -selfmaps, other than the fact that they exist, are that they are asymptotically unique and asymptotically central:

Proposition 8.3. [HS98, Corollary 3.7] *If f and g are two v_n -selfmaps of X then $f^i = g^j$ for some $i, j \geq 1$.*

Proposition 8.4. [HS98, Corollary 3.8] *If f is a v_n -selfmap of X then some power of f is contained in the center of $[X, X]_\bullet$.*

For our purposes, we need to include the following result:

Lemma 8.5. *If f is a v_n -selfmap of X then some power of f is \otimes -balanced.*

Proof. This is straightforward in light of Proposition 8.3 by recognizing that $f \otimes X$ and $X \otimes f$ are two v_n -selfmaps of $X \otimes X$. \square

The notion of a v_n -selfmap leads to a very complete description of the centers of graded endomorphism rings of finite p -local spectra—up to nilpotents.

Theorem 8.6. [HS98, Corollary 5.5, Proposition 5.6] *Let X be a finite p -local spectrum and let f be a graded endomorphism of X which is in the center of $[X, X]_\bullet$. If f is degree 0 then some power of f is a multiple of the identity; otherwise, f is nilpotent or a v_n -selfmap.*

Corollary 8.7. *Let $n \geq 1$ and $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$. The space $\mathrm{Spec}^h(A_X^\bullet)$ consists of two points: a generic point consisting of the homogeneous nilpotents and a closed point consisting of the homogeneous non-units. The closed point is of the form $\sqrt{(f)}$ for any \otimes -balanced v_n -selfmap $f : \Sigma^d X \rightarrow X$.*

Proof. If $f \in A_X^0$ then $f^k = m \cdot \text{id}_X$ for some $k \geq 1$ and $m \in \mathbb{Z}$. If $p \mid m$ then it follows from the Nilpotence Theorem and the fact that $X \in \mathcal{C}_1$ that f is nilpotent, while if $p \nmid m$ then $m \cdot \text{id}_X$ is an isomorphism. Thus, every element of degree zero in A_X^\bullet is either nilpotent or a unit. On the other hand, every element of non-zero degree is either nilpotent or a v_n -selfmap; moreover, v_n -selfmaps are not nilpotent since $X \notin \mathcal{C}_{n+1}$, nor are they units. It follows that the homogeneous non-units form an ideal \mathfrak{m} which is necessarily the unique maximal homogeneous ideal of A_X^\bullet . On the other hand, the ideal of homogeneous nilpotents \mathfrak{n} is readily seen to be prime just using the fact that the product of two v_n -selfmaps is again a v_n -selfmap and the product of a unit and a v_n -selfmap is again a v_n -selfmap. Since a v_n -selfmap exists in A_X^\bullet (by Lemma 8.5 and Theorem 8.2), $\mathfrak{n} \subsetneq \mathfrak{m}$. Moreover, if \mathfrak{p} is a homogeneous prime ideal then $\mathfrak{n} \subsetneq \mathfrak{p}$ implies that there is a v_n -selfmap f contained in \mathfrak{p} . By the asymptotic uniqueness of v_n -selfmaps, it follows that every v_n -selfmap is contained in \mathfrak{p} , so that $\mathfrak{p} = \mathfrak{m}$. We conclude that the only homogeneous primes of A_X^\bullet are \mathfrak{n} and \mathfrak{m} . Finally note that the asymptotic uniqueness of v_n -selfmaps implies that $\mathfrak{m} = \sqrt{\langle f \rangle}$ for any \otimes -balanced v_n -selfmap $f : \Sigma^d X \rightarrow X$. \square

Lemma 8.8. *Let $n \geq 1$ and $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$. If $f : \Sigma^d X \rightarrow X$ is a v_n -selfmap then $\text{cone}(f) \in \mathcal{C}_{n+1} \setminus \mathcal{C}_{n+2}$.*

Proof. This is a routine exercise using the long exact sequences obtained by applying Morava K -theories to an exact triangle for $f : \Sigma^d X \rightarrow X$. \square

We are now in a position to examine the structure of SH^{fin} via higher comparison maps. The starting point is the comparison map for the unit object: the sphere spectrum. It is well-known that the endomorphism ring of the sphere spectrum is $\text{End}_{\text{SH}^{\text{fin}}}(\mathbb{1}) \cong \mathbb{Z}$. On the other hand, $\pi_i(\mathbb{1}) = 0$ for $i < 0$ and all the graded endomorphisms of positive degree are nilpotent by Nishida's theorem. It follows that $\text{Spec}^h(\text{End}_{\text{SH}^{\text{fin}}}^\bullet(\mathbb{1})) \simeq \text{Spec}(\text{End}_{\text{SH}^{\text{fin}}}(\mathbb{1}))$ and that the graded and ungraded comparison maps coincide. Moreover, triangular localization with respect to $\mathfrak{p}^\bullet \subset \text{End}_{\text{SH}^{\text{fin}}}^\bullet(\mathbb{1})$ is the same as triangular localization with respect to $\mathfrak{p}^0 \subset \text{End}_{\text{SH}^{\text{fin}}}(\mathbb{1})$.

Proposition 8.9. *The triangular localization of SH^{fin} at a prime $(p) \subset \text{End}_{\text{SH}^{\text{fin}}}(\mathbb{1})$ is equivalent as a tensor triangulated category to the stable homotopy category of finite p -local spectra $\text{SH}_{(p)}^{\text{fin}}$ while the triangular localization of SH^{fin} at the prime (0) is equivalent to the quotient of SH^{fin} by the finite torsion spectra $\text{SH}_{\text{tor}}^{\text{fin}}$.*

Proof. Proving the first statement amounts to showing that the thick \otimes -ideal $\langle \text{cone}(d : \mathbb{1} \rightarrow \mathbb{1}) \mid p \nmid d \rangle$ is precisely the set of finite p -acyclic spectra. One inclusion is easily obtained by applying $\pi_\bullet(-) \otimes \mathbb{Z}_{(p)}$ to an exact triangle for $d : \mathbb{1} \rightarrow \mathbb{1}$. On the other hand, if X is a finite p -acyclic spectrum then $\pi_i(X)$ is finite with no p -torsion for all $i \in \mathbb{Z}$. For any finite spectrum X it is straightforward to check that $\mathcal{I}_X := \{Y \in \text{SH}^{\text{fin}} \mid [Y, X]_i \text{ is finite with no } p\text{-torsion for all } i \in \mathbb{Z}\}$ is a thick subcategory of SH^{fin} . If X is finite p -acyclic then \mathcal{I}_X contains $\mathbb{1}$ and hence contains the whole of SH^{fin} . In particular \mathcal{I}_X contains X itself and we conclude that id_X has finite order d prime to p . Then $\Sigma X \oplus X \simeq \text{cone}(d \cdot \text{id}_X) \simeq X \otimes \text{cone}(d \cdot \text{id}_{\mathbb{1}})$ establishes that X is contained in $\langle \text{cone}(d : \mathbb{1} \rightarrow \mathbb{1}) \mid p \nmid d \rangle$. A similar approach can be used to prove that $\text{SH}_{\text{tor}}^{\text{fin}}$ is equal to $\langle \text{cone}(d : \mathbb{1} \rightarrow \mathbb{1}) \mid d \neq 0 \rangle$. \square

Then consider the map $\rho_{\mathbb{1}} : \mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}) \rightarrow \mathrm{Spec}(\mathbb{Z})$. Triangular localization with respect to the generic point $(0) \in \mathrm{Spec}(\mathbb{Z})$ gives a map

$$\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}/\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}) \rightarrow \mathrm{Spec}(\mathbb{Q})$$

and we conclude that the fiber over (0) is $V(\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}) \subset \mathrm{SH}^{\mathrm{fin}}$. In fact, one can show that $\mathrm{SH}^{\mathrm{fin}}/\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}} \simeq D^b(\mathbb{Q})$ (see [Mar83, page 113]) and hence the spectrum of $\mathrm{SH}^{\mathrm{fin}}/\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}$ is a single point. Moreover, $\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}$ itself is prime and so we conclude that the fiber over (0) is the single point $\{\mathrm{SH}_{\mathrm{tor}}^{\mathrm{fin}}\}$.

Next consider the fiber over a closed point $(p) \in \mathrm{Spec}(\mathbb{Z})$. Triangular localization provides a map $\mathrm{Spc}(\mathrm{SH}_{(p)}^{\mathrm{fin}}) \rightarrow \mathrm{Spec}(\mathbb{Z}_{(p)})$ and the fiber over the unique closed point $(p) \in \mathrm{Spec}(\mathbb{Z}_{(p)})$ is $\mathrm{supp}(\mathrm{cone}(p.\mathrm{id}_{\mathbb{1}})) = \overline{\{\mathcal{C}_2\}}$. In other words, the fiber includes everything with the exception of a single point: \mathcal{C}_1 . The next step is to define $X_1 := \mathrm{cone}(p.\mathrm{id}_{\mathbb{1}})$ and consider

$$\rho_{X_1, A_{X_1}^{\bullet}} : \overline{\{\mathcal{C}_2\}} = \mathrm{supp}(X_1) \rightarrow \mathrm{Spec}^h(A_{X_1}^{\bullet}).$$

By Corollary 8.7 the unique closed point of $\mathrm{Spec}^h(A_{X_1}^{\bullet})$ is of the form $\sqrt{(f)}$ for any \otimes -balanced v_n -selfmap f of X_1 and by Lemma 8.8 the fiber over this point is $\mathrm{supp}(\mathrm{cone}(f)) = \overline{\{\mathcal{C}_3\}}$. Again the fiber consists of everything except for one point and the process continues. At the n th step we have an object X_n and a map

$$\rho_{X_n, A_{X_n}^{\bullet}} : \overline{\{\mathcal{C}_{n+1}\}} = \mathrm{supp}(X_n) \rightarrow \mathrm{Spec}^h(A_{X_n}^{\bullet}).$$

The unique closed point is generated as a radical ideal by any \otimes -balanced v_n -selfmap f_n and the fiber over this point is $\mathrm{supp}(\mathrm{cone}(f_n)) = \overline{\{\mathcal{C}_{n+2}\}}$. Altogether this gives a filtration of the fiber

$$\rho_{\mathrm{SH}_{(p)}^{\mathrm{fin}}, \mathbb{1}}^{-1}((p)) = \overline{\{\mathcal{C}_2\}} \supset \overline{\{\mathcal{C}_3\}} \supset \overline{\{\mathcal{C}_4\}} \supset \cdots \quad (8.10)$$

where exactly one point is removed at each step. All of this may be better appreciated by considering the picture of $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$ displayed on page 4. Triangular localization at p focuses on a single branch and then each successive comparison map chops off the root heading towards $\mathcal{C}_{p, \infty}$. Note that we obtain every irreducible closed subset of the fiber (8.10) except for the closed point $\overline{\{\mathcal{C}_{\infty}\}} = \{\mathcal{C}_{\infty}\}$. The fact that this point is missed shouldn't be alarming since it corresponds to an irreducible closed subset which is not Thomason. If all the rings involved are noetherian then we can only expect to obtain Thomason closed subsets since our comparison maps are spectral—when using these strategies we should take arbitrary intersections of all of the closed subsets that we obtain. Keep in mind that the Thomason closed subsets are a basis of closed sets, so if we can obtain all the Thomason closed subsets of the spectrum then we have obtained the entire spectrum.

REFERENCES

- [Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.
- [Bal07] Paul Balmer. Supports and filtrations in algebraic geometry and modular representation theory. *Amer. J. Math.*, 129(5):1227–1250, 2007.
- [Bal10a] Paul Balmer. Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010.
- [Bal10b] Paul Balmer. Tensor triangular geometry. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 85–112, New Delhi, 2010. Hindustan Book Agency.

- [BKS07] Aslak Bakke Buan, Henning Krause, and Øyvind Solberg. Support varieties: an ideal approach. *Homology, Homotopy Appl.*, 9(1):45–74, 2007.
- [DS] Ivo Dell’Ambrogio and Greg Stevenson. Even more spectra: tensor triangular comparison maps via graded commutative 2-rings. *Appl. Categ. Structures*. To appear.
- [Hoc69] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.
- [HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.
- [HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.
- [Kra10] Henning Krause. Localization theory for triangulated categories. In *Triangulated categories*, volume 375 of *London Math. Soc. Lecture Note Ser.*, pages 161–235. Cambridge Univ. Press, Cambridge, 2010.
- [KY11] Henning Krause and Yu Ye. On the centre of a triangulated category. *Proc. Edinb. Math. Soc. (2)*, 54(2):443–466, 2011.
- [Li12] Huishi Li. On monoid graded local rings. *J. Pure Appl. Algebra*, 216(12):2697–2708, 2012.
- [LS] Yu-Han Liu and Susan J. Sierra. Prime spectra of derived quiver representations. *Comm. Algebra*. To appear.
- [Mar83] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983.
- [Sta] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu>.
- [Tho97] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.